ON THE FINITENESS OF STRONG MAXIMAL FUNCTIONS ASSOCIATED TO FUNCTIONS WHOSE INTEGRALS ARE STRONGLY DIFFERENTIABLE

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Abstract. Besicovitch proved that if f is an integrable function on $\mathbb{R}^2$ whose associated strong maximal function $M_S f$ is finite a.e., then the integral of f is strongly differentiable. On the other hand, Papoulis proved the existence of an integrable function on $\mathbb{R}^2$ (taking on both positive and negative values) whose integral is strongly differentiable but whose associated strong maximal function is infinite on a set of positive measure. In this paper, we prove that if $n \geq 2$ and if f is a measurable *nonnegative* function on $\mathbb{R}^n$ whose integral is strongly differentiable and moreover such that $f(1+\log^+ f)^{n-2}$ is integrable, then $M_S f$ is finite a.e. We also show this result is sharp by proving that, if φ is a continuous increasing function on $[0,\infty)$ such that $\varphi(0)=0$ and with $\varphi(u) = o(u(1+\log^+ u)^{n-2})$ $(u \to \infty)$, then there exists a nonnegative measurable function f on $\mathbb{R}^n$ such that $\varphi(f)$ is integrable on $\mathbb{R}^n$ and the integral of f is strongly differentiable, although $M_S f$ is infinite almost everywhere.

1. Introduction

One of the foundational results of modern analysis, the *Lebesgue Differentiation Theorem* tells us that, if f is an integrable function on $\mathbb{R}^n$, then for a.e. $x \in \mathbb{R}^n$ we have

$$\lim_{k\to\infty} \frac{1}{|B_k|} \int_{B_k} f \;=\; f(x)$$

holds for every sequence $\{B_k\}$ of balls containing x whose diameters tend to 0. This result does not necessarily hold, however, if we replace balls by more general convex sets. For example, Saks proved in [9] that there exists a function $f \in L^1(\mathbb{R}^2)$ such that, for a.e. $x \in \mathbb{R}^2$, there exists a sequence $\{R_k\}$ of rectangles with sides parallel to the coordinate axes containing x whose diameters tend to 0 for which

$$\lim_{k\to\infty} \frac{1}{|R_k|} \int_{R_k} f \;= \infty\,.$$

However, Jessen, Marcinkiewicz, and Zygmund proved in [5] that if a measurable function f on $\mathbb{R}^2$ satisfies the more stringent size condition that $f(1+\log^+|f|)$ is integrable, then for

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a.e. $x \in \mathbb{R}^2$ we have

$$\lim_{k\to\infty} \frac{1}{|R_k|} \int_{R_k} f \ = f(x) \tag{1.1}$$

holds for every sequence $\{R_k\}$ of rectangles with sides parallel to the coordinate axes containing x whose diameters tend to 0.

We briefly recall some useful definitions and terminology. If for a given function f on $\mathbb{R}^n$ the limit (1.1) holds for a.e. $x \in \mathbb{R}^n$, we say that the integral of f is *strongly differentiable*, frequently abbreviated $\int f$ is strongly differentiable. If this limit holds for every x in a set E, we say that $\int f$ is *strongly differentiable on the set* E. Also, given a measurable function f on $\mathbb{R}^n$, the *strong maximal function* of f is denoted by $M_S f$ and defined on $\mathbb{R}^n$ by

$$M_S f(x) = \sup_{x\in R\in\mathcal{B}_n} \frac{1}{|R|} \int_R |f| \ ,$$

where $\mathcal{B}_n$ is defined to be the collection of open rectangular parallelepipeds in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes. (Note this definition holds for any measurable f, keeping in mind that the integrals over R above may take on infinite value.)

In [1] Besicovitch generalized the result of Jessen, Marcinkiewicz, and Zygmund by proving the following:

Theorem 1 (Besicovitch)**.** *Let f be an integrable function on $\mathbb{R}^2$ whose associated strong maximal function $M_S f$ is finite a.e. Then $\int f$ is strongly differentiable.*

The multidimensional version of Theorem 1 was proved by Ward [11]. Extensions for certain general classes of differentiation bases were suggested by de Guzmán and Menárguez [3] (see Section IV.3) and Oniani [6, 7]. Hagelstein, Herden, and Stokolos proved in [4] an analogue of Theorem 1 for ergodic means.

Besicovitch's theorem provides a generalization of the result of Jessen, Marcinkiewicz, and Zygmund as the strong maximal operator M_S satisfies the *weak type estimate* [1]

$$|\{x \in \mathbb{R}^n : M_S f(x) > \alpha\}| \le C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha} \left(1 + \log^+ \frac{|f|}{\alpha}\right)^{n-1} . \tag{1.2}$$

It is natural to consider whether Besicovitch's Theorem has a converse, namely, if $f \in L^1(\mathbb{R}^n)$ is such that $\int f$ is strongly differentiable, must $M_S f$ be finite a.e.? Note that in the one-dimensional case this does hold but the situation is somewhat misleading, as on $\mathbb{R}^1$ the strong maximal operator M_S agrees with the Hardy-Littlewood maximal operator which satisfies a weak type $(1,1)$ inequality. It is the case of dimensions $n \ge 2$ that the problem

[1] This estimate, frequently known as the *Jessen-Marcinkiewicz-Zygmund inequality*, seems to first appear in the paper [2] of Fava.

becomes interesting.

In general, the converse does not hold, as was first shown by the following result [8] of Papoulis:

Theorem 2 (Papoulis)**.** *There exists an integrable function f on $\mathbb{R}^2$ such that $\int f$ is strongly differentiable but for which $M_S f$ is infinite on a set of positive measure.*

Of particular interest here is the fact that the function f constructed by Papoulis is such that $\int |f|$ is *not* strongly differentiable. This invites the following problem: if f is a nonnegative integrable function on $\mathbb{R}^2$ whose integral is strongly differentiable, must $M_S f$ be finite a.e.? The purpose of this paper is to prove that this is indeed the case, and moreover we have the following.

Theorem 3. *Let $n \geq 2$ and f be a measurable nonnegative function on $\mathbb{R}^n$ whose integral is strongly differentiable. If $f(1+\log^+ f)^{n-2}$ is integrable on $\mathbb{R}^n$, then $M_S f$ is finite a.e.*

Moreover, we show this result is sharp in the sense that it does not hold if we replace the $u(1+\log^+ u)^{n-2}$ condition by a weaker one:

Theorem 4. *Let $n \geq 2$, and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous increasing function with $\varphi(0) = 0$ satisfying the condition $\varphi(u) = o(u(1+\log^+ u)^{n-2})$ $(u \to \infty)$. Then there exists a nonnegative function f on $\mathbb{R}^n$ so that $\varphi(f)$ is integrable on $\mathbb{R}^n$ and $\int f$ is strongly differentiable, although $M_S f$ is infinite almost everywhere.*

Note that a nonnegative function $f \in L^1(\mathbb{R}^3)$ with strongly differentiable integral for which $M_S f$ is infinite on a set of positive measure was constructed by Zerekidze [12].

Note also that the integral of every function f on $\mathbb{R}^n$ $(n \geq 2)$ of the type $f(x,y) = g(x)\chi_{[0,1]^{n-1}}(y)$ $(x \in \mathbb{R}, y \in \mathbb{R}^{n-1})$, where $g \in L^1(\mathbb{R})$, is strongly differentiable. This easily implies that there exist nonnegative functions f on $\mathbb{R}^n$ with strongly differentiable integrals for which $f(1+\log^+ f)^{n-2}$ is integrable but $f(1+\log^+ f)^{n-1}$ not.

In the second section of this paper we provide a proof of Theorem 3; in the third section we provide a proof of Theorem 4.

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2. Proof of Theorem 3

Suppose f is a measurable nonnegative function on $\mathbb{R}^n$, $\int f$ is strongly differentiable, and $M_S f$ is infinite on a set of positive measure. It suffices to show that

$$\int_{\mathbb{R}^n} f\left(1+\log^+ f\right)^{n-2} \;=\; \infty\,.$$

We assume without loss of generality that f is finite a.e. For $1 \le j \le n$, we define the maximal operator $M_{S,j}$ on measurable functions on $\mathbb{R}^n$ by

$$M_{S,j}g(x) = \sup_{x\in R} \frac{1}{|R|}\int_R |g| \ ,$$

where the supremum is over all parallelepipeds R in $\mathcal{B}_n$ containing x whose longest side lies in the j-th coordinate direction. Since $M_S f(x) = \max_{1\le j\le n} M_{S,j}f(x)$, there exists $1 \le j \le n$ so that $M_{S,j}f$ is infinite on a set E of positive measure in $\mathbb{R}^n$. Without loss of generality we assume $j = 1$. Now, taking into account that f is measurable and finite a.e., we can find $0 < c < \infty$ for which $f(x) \le c$ on a set $E' \subset E$ of positive measure. Since $\int f$ is strongly differentiable, there exists a set $E'' \subset E'$ with $|E' \setminus E''| = 0$ such that for every $x \in E''$, $\frac{1}{|R|}\int_R f \to f(x)$ as $x \in R \in \mathcal{B}_n$ and diam $R \to 0$. Hence, for every $x \in E''$ there exists $k_x \in \mathbb{N}$ for which

$$\sup_{x\in R\in\mathcal{B}_n, \mathrm{diam} R<1/k_x} \frac{1}{|R|}\int_R f \le c+1.$$

Note that for every $k \in \mathbb{N}$ the function

$$g_k(x) = \sup_{x\in R\in\mathcal{B}_n, \mathrm{diam} R<1/k} \frac{1}{|R|}\int_R f$$

is measurable, and the sequence of the functions g_k $(k \in \mathbb{N})$ is nonincreasing. This implies that the sequence of the measurable sets $E_k = \{x \in E'' : g_k(x) \le c+1\}$ $(k \in \mathbb{N})$ is nondecreasing. Taking into account the equality $\cup_{k=1}^{\infty} E_k = E''$, we find $k \in \mathbb{N}$ for which E_k is of positive measure. Now setting $F = E_k$, $C = c+1$ and $\varepsilon = 1/(n^{1/2}k)$, we conclude the validity of the following claim: There exist a set F with positive measure and numbers $C > 1$ and $\epsilon > 0$ such that $M_{S,1}f(x) = \infty$ for every $x \in F$ and $\frac{1}{|R|}\int_R f < C$ for every rectangular parallelepiped R in $\mathcal{B}_n$ containing some point x from F and with diameter less than $n^{1/2}\epsilon$. Clearly, there exists $t \in \mathbb{R}$ such that $(\{t\} \times \mathbb{R}^{n-1}) \cap F$ is of positive $\mathcal{H}^{n-1}$ measure. Denote $A = (\{t\} \times \mathbb{R}^{n-1}) \cap F$. Without loss of generality, we assume $t = 0$.

We now introduce a rearrangement f^* of f along lines parallel to the first coordinate axis. In particular, we define $f^*(t, y)$ such that, for fixed $y \in \mathbb{R}^{n-1}$, $f^*(\cdot, y)$ and $f(\cdot, y)$ are equimeasurable, $f^*(t, y) = 0$ if $t \le 0$, and $f^*(t_1, y) \ge f^*(t_2, y)$ whenever $0 < t_1 \le t_2$. Note that f^* is supported on the "right-half" of $\mathbb{R}^n$ and such that on that right half f^* is nonincreasing along lines parallel to the x_1-axis.

Note that as f and f^* are equimeasurable, we have

$$\int_{\mathbb{R}^n} f^*(1+\log^+ f^*)^{n-2} \ = \ \int_{\mathbb{R}^n} f(1+\log^+ f)^{n-2} \ .$$

For $y \in \mathbb{R}^{n-1}$ we set

$$\bar{f}(y) = \frac{1}{\epsilon}\int_0^{\epsilon} f^*(t, y)\, dt \ .$$

Let $\gamma > C$. Since $\frac{1}{|R|}\int_R f < C$ for every $R \in \mathcal{B}_n$ of diameter less than $n^{1/2}\epsilon$ and containing some point from A, for every $(0, y) \in A$ there must exist $R = (a, b) \times R' \in \mathcal{B}_n$ such that

$a < 0 < b, b - a \geq \varepsilon, R' \in \mathcal{B}_{n-1}, y \in R'$ and

$$\frac{1}{|(a,b) \times R'|} \int_{(a,b)\times R'} f > \gamma.$$

(Note that, as a matter of technique, it is at this point that we have used that $M_{S,1}f$ is infinite on a set of positive measure. Note this condition has enabled us to construct a set A of $\mathcal{H}^{n-1}$ positive measure lying in a section of $\mathbb{R}^n$ orthogonal to the x_1 axis and $\epsilon > 0$ so that, for every $\gamma > 0$, we have every point in A is contained in a rectangular parallelepiped R with x_1-length no less than ϵ so that the average of f over R exceeds γ.)

Hence, by the definition of f^* it is easy to see that

$$\frac{1}{|(0,\varepsilon) \times R'|} \int_{(0,\varepsilon)\times R'} f^* \geq \frac{1}{|(a,b) \times R'|} \int_{(a,b)\times R'} f > \gamma.$$

Then we have

$$\frac{1}{|R'|_{n-1}} \int_{R'} \bar{f} = \frac{1}{|(0,\varepsilon) \times R'|} \int_{(0,\varepsilon)\times R'} f^* > \gamma.$$

Thus for every $(0, y) \in A$ we have the equality $M_{S,\mathbb{R}^{n-1}}\bar{f}(x) > \gamma$. (Here and below for clearness we use the notation $M_{S,\mathbb{R}^{n-1}}$ for the strong maximal operator related with $\mathbb{R}^{n-1}$.) Consequently, by the weak type estimate (1.2) for $M_{S,\mathbb{R}^{n-1}}\bar{f}$, we obtain

$$\mathcal{H}^{n-1}(A) \leq C_{n-1} \int_{\mathbb{R}^{n-1}} \frac{\bar{f}}{\gamma} \left(1 + \log^+ \frac{\bar{f}}{\gamma}\right)^{n-2} ,$$

and hence

$$\begin{aligned}
\gamma C_{n-1}^{-1} \mathcal{H}^{n-1}(A) &\leq \int_{\mathbb{R}^{n-1}} \bar{f} \left(1 + \log^+ \bar{f}\right)^{n-2} \\
&\leq \frac{1}{\epsilon} \int_{(0,\epsilon)\times\mathbb{R}^{n-1}} f^* \left(1 + \log^+ f^*\right)^{n-2} \qquad \text{(by Jensen's Inequality)} \\
&\leq \frac{1}{\epsilon} \int_{\mathbb{R}^n} f \left(1 + \log^+ f\right)^{n-2} .
\end{aligned}$$

Note that γ can be arbitrarily large in the last estimate. Hence

$$\int_{\mathbb{R}^n} f \left(1 + \log^+ f\right)^{n-2} = \infty ,$$

as desired.

3. Proof of Theorem 4

We first consider the case that $n = 2$. Here we have that $\varphi(u) = o(u)$, and hence there exists a function h on $\mathbb{R}$ supported and nonincreasing on $(0, 1)$ such that $h \notin L^1(\mathbb{R})$ although $\varphi(h)$ is integrable. Setting $f(x_1, x_2) = h(x_1)\chi_{[0,1]\times[0,1]}(x_1, x_2)$ provides the desired example, since f is strongly differentiable a.e. and $\varphi(f)$ is integrable on $\mathbb{R}^2$, although $M_S f$ is infinite

on $\mathbb{R}^2$.

For the remainder of the proof we assume $n > 2$.

From the paper [10] of Saks (namely, see Theorem A in [10]), we know there exists a nonnegative function $h \in L^1(\mathbb{R}^{n-1})$ with $\{h \neq 0\} \subset [0,1]^{n-1}$ such that $\varphi(h)$ is integrable and $M_{S,\mathbb{R}^{n-1}}h$ is infinite a.e. on $[0,1]^{n-1}$. From h we may construct a function $g \in L^1(\mathbb{R}^{n-1})$ such that $\varphi(g)$ is integrable and $M_{S,\mathbb{R}^{n-1}}g$ is infinite a.e. on $\mathbb{R}^{n-1}$. This can be done, for instance, by setting

$$g(x_1,\dots,x_{n-1}) = \sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_{n-1}=\infty}^{\infty} c_{j_1,\dots,j_{n-1}} h(x_1 - j_1,\dots,x_{n-1} - j_{n-1}) \ ,$$

defining the constants $c_{j_1,\dots,j_{n-1}}$ to be in $(0,1)$ and such that $\varphi(g)$ is integrable. This holds provided

$$\sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_{n-1}=\infty}^{\infty} \|\varphi(c_{j_1,\dots,j_{n-1}} h)\|_{L^1(\mathbb{R}^{n-1})} < \infty$$

and can be achieved by setting the $c_{j_1,\dots,j_{n-1}}$ to satisfy (say)

$$\|\varphi(c_{j_1,\dots,j_{n-1}} h)\|_{L^1(\mathbb{R}^{n-1})} < 2^{-(|j_1|+\cdots+|j_{n-1}|)} \ .$$

That this may be done can be seen by the Lebesgue Dominated Convergence Theorem, taking advantage of the fact that $\lim_{u\to 0^+} \varphi(u) = 0$.

Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence of nonnegative real numbers such that, if we define the sets $E_k \subset \mathbb{R}^{n-1}$ by

$$E_k = \{a_k < g \leq a_{k+1}\} \ ,$$

we have

$$|\{M_{S,\mathbb{R}^{n-1}}(g\chi_{E_k}) > 2^k\} \cap [-k,k]^{n-1}|_{n-1} \geq (2k)^{n-1} - 2^{-k} \ .$$

Define the function f on $\mathbb{R}^n$ by

$$f(x_1,\dots,x_n) = \sum_{k=1}^{\infty} \chi_{(k-1,k)}(x_1) g(x_2,\dots,x_n) \chi_{E_k}(x_2,\dots,x_n) \ .$$

Observe that, by the Fubini Theorem, we have $\varphi(f)$ is integrable on $\mathbb{R}^n$. Moreover, as f is bounded on any strip $(-t,t) \times \mathbb{R}^{n-1}$ $(t > 0)$, we have that $\int f$ is strongly differentiable.

It remains to show that $M_S f$ is infinite a.e. on $\mathbb{R}^n$. To do this, it suffices to show that, if $j \geq 1$, then $M_S f$ is infinite a.e. on the cube $[-j,j]^n$. Note that if $k \geq j$, we have

$$\left|\left\{x \in [-j,j]^n : M_S f(x) > \frac{2^k}{2k}\right\}\right| \geq (2j)^n - 2j \cdot 2^{-k}$$

as can be seen by averaging f over rectangular parallelepipeds of the form $(-k,k)\times R$, where $R\in\mathcal{B}_{n-1}$. Letting k tend to infinity, we see

$$|\{x\in[-j,j]^n : M_S f(x)=\infty\}| = (2j)^n\ ,$$

and hence $M_S f$ is infinite a.e. on the cube $[-j,j]^n$, as desired.

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