$L^p(\mathbb{R}^2)$ BOUNDS FOR GEOMETRIC MAXIMAL OPERATORS ASSOCIATED TO HOMOTHECY INVARIANT CONVEX BASES

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Abstract. Let $\mathcal{B}$ be a nonempty homothecy invariant collection of convex sets of positive finite measure in $\mathbb{R}^2$. Let $M_{\mathcal{B}}$ be the geometric maximal operator defined by

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| \; .$$

We show that either $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^2)$ for every $1 < p \leq \infty$ or that $M_{\mathcal{B}}$ is unbounded on $L^p(\mathbb{R}^2)$ for every $1 \leq p < \infty$. As a corollary, we have that any density basis that is a homothecy invariant collection of convex sets in $\mathbb{R}^2$ must differentiate $L^p(\mathbb{R}^2)$ for every $1 < p \leq \infty$.

1. Introduction

Let $\mathcal{B}$ denote a nonempty collection of sets of positive measure in $\mathbb{R}^n$. Associated to $\mathcal{B}$ is the *geometric maximal operator* $M_{\mathcal{B}}$ defined on measurable functions f on $\mathbb{R}^n$ by

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| \; ,$$

where the supremum is over all members R in $\mathcal{B}$ containing x. If $\mathcal{B}$ is the collection of all balls in $\mathbb{R}^n$, then $M_{\mathcal{B}}$ is the well-known *Hardy-Littlewood maximal operator* M_{HL}. If $\mathcal{B}$ is the collection of all rectangular parallelepipeds in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes, then $M_{\mathcal{B}}$ is the *strong maximal operator* M_S. If $\mathcal{B}$ is the collection of all rectangles in $\mathbb{R}^2$ whose longest sides have slope of the form 2^{-k} for some natural number k, then $M_{\mathcal{B}}$ is the *lacunary maximal operator* M_{lac}. If $\mathcal{B}$ is the collection of all rectangular parallelepipeds in $\mathbb{R}^n$, then $M_{\mathcal{B}}$ is the *Kakeya-Nikodym maximal operator* which we denote here by M_{KN}.

The L^p boundedness properties of these particular maximal operators are now well understood. M_{HL} and M_S are bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$, M_{lac} is bounded on $L^p(\mathbb{R}^2)$ for $1 < p \leq \infty$, but M_{KN} is unbounded on $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$. For proofs of these results we refer the reader to [14].

$M_{\mathcal{B}_\Omega}$ is said to be a *directional maximal operator* if there is a nonempty set of directions Ω for which $\mathcal{B}_\Omega$ consists of every rectangle oriented in one of the directions in Ω. Bateman

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proved in [1] that if $M_{\mathcal{B}_\Omega}$ is a directional maximal operator associated to directions in the plane, then either $M_{\mathcal{B}_\Omega}$ is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \leq \infty$ or $M_{\mathcal{B}_\Omega}$ is unbounded on $L^p(\mathbb{R}^2)$ for all $1 \leq p < \infty$. This type of dichotomy for directional maximal operators on the plane was in many respects anticipated by earlier papers including [2] and [15]. Such an explicit dichotomy is still unknown for directional maximal operators acting on functions on $\mathbb{R}^n$ for $n \geq 3$, although progress has been made on this problem by, among others, Parcet and Rogers [12] and Kroc and Pramanik [8].

It is natural to consider the question of, if $\mathcal{B}$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ (not necessarily consisting of *all* rectangles or rectangular parallelepipeds oriented in a certain set of directions), whether $M_{\mathcal{B}}$ must be bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq \infty$ or whether it must be unbounded on $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$. This is a difficult problem. The purpose of this paper is to provide a solution in the two-dimensional case. Our solution builds on ideas accumulated over the past four decades associated to the behavior of maximal operators associated to a lacunary set of directions, the Besicovitch and Kakeya conjectures, Bernouilli percolation, and sticky maps. Particularly relevant to our solution are papers of Nagel, Stein, and Wainger [11]; Sjögren and Sjölin [13]; Lyons [10]; Katz [7]; Katz, Łaba, and Tao [6]; Bateman and Katz [2]; and Bateman [1].

Theorem 1. *Let $\mathcal{B}$ be a nonempty homothecy invariant collection of convex sets of positive finite measure in $\mathbb{R}^2$. Let $M_{\mathcal{B}}$ be the geometric maximal operator defined by*

$$M_{\mathcal{B}}f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| \ .$$

Either $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^2)$ for every $1 < p \leq \infty$ or $M_{\mathcal{B}}$ is unbounded on $L^p(\mathbb{R}^2)$ for every $1 \leq p < \infty$.

An immediate application of this theorem falls in the classical topic of differentiation of integrals. Let $\mathcal{B}$ be a homothecy invariant collection of bounded sets in $\mathbb{R}^2$ of positive measure. Recall that $\mathcal{B}$ is said to be a *density basis* provided, given a measurable set $E \subset \mathbb{R}^2$, for a.e. x we have $\lim_{k\to\infty} \frac{1}{|R_k|}\int_{R_k} \chi_E = \chi_E(x)$ whenever $\{R_k\}$ is a sequence of sets in $\mathcal{B}$ containing x whose diameters are tending to 0. Hagelstein and Stokolos proved in [5] that a density basis consisting of convex sets in $\mathbb{R}^2$ must differentiate $L^p(\mathbb{R}^2)$ for sufficiently large p. As a corollary of Theorem 1, we then have the following.

Corollary 1. *Let $\mathcal{B}$ be a homothecy invariant collection of convex sets of positive finite measure in $\mathbb{R}^2$. If $\mathcal{B}$ is a density basis, then $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^2)$ for $1 < p \leq \infty$ and $\mathcal{B}$ must differentiate $L^p(\mathbb{R}^2)$ for $1 < p \leq \infty$.*

It is worth noting that the Hagelstein-Stokolos proof in [5] that a homothecy invariant density basis consisting of convex sets must differentiate $L^p(\mathbb{R}^n)$ for sufficiently large p falls in the classical realm of differentiation of integrals, relying on delicate arguments involving covering lemmas and the Calderón-Zygmund decomposition. The proof of Corollary 1, however, relies on not only these classical ideas in differentiation theory but also techniques that are Fourier analytic in nature. In many respects this illustrates a paradigm in contemporary harmonic analysis that L^p bounds for maximal operators such as the lacunary maximal operator should in theory be provable via covering lemmas (see A. Córdoba and R. Fefferman [3]), but at the moment the best known bounds (in particular in the range $1 < p \leq 2$) have been attained only via Fourier analytic arguments involving Littlewood-Paley theory, square functions, and complex interpolation.

The remainder of this paper will be devoted to a proof of Theorem 1. Our proof will rely heavily on the ideas of Bateman and Katz [1, 2] related to recognizing the fact that the boundedness of a directional maximal operator $M_{\mathcal{B}_\Omega}$ on $L^p(\mathbb{R}^2)$ corresponds to the ability to cover Ω by finitely many N-lacunary sets; we also take advantage of recent ideas of Gauvan [4] that enable us to modify the Bateman-Katz methodology to accommodate scenarios when $\mathcal{B}$ is a homothecy invariant basis of convex sets not necessarily corresponding to a directional maximal operator.

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2. Proof of Theorem 1

Proof of Theorem 1. If $S \subset \mathbb{R}^2$, and $x \in \mathbb{R}^2$, we define the translate $\tau_x S$ of S to be the set in $\mathbb{R}^2$ satisfying

$$\chi_{\tau_x S}(y) = \chi_S(y - x)\ .$$

If $r > 0$, we define the dilate rS of S to be the set in $\mathbb{R}^2$ satisfying

$$\chi_{rS}(y) = \chi_S(y/r)\ .$$

Recall that if $\mathcal{B}$ is a homothecy invariant basis of sets in $\mathbb{R}^2$, then for any $R \in \mathcal{B}$, $r > 0$, and $x \in \mathbb{R}^2$ we have $\tau_x R \in \mathcal{B}$ and $rR \in \mathcal{B}$.

Now, *since the members of $\mathcal{B}$ are convex*, by a result of Lassak [9] we have that if $S \in \mathcal{B}$ then there exists a rectangle R containing S such that a translate of a $\frac{1}{2}$-fold dilate of R is contained in S. Hence we may assume without loss of generality that every set $S \in \mathcal{B}$ is covered by a rectangle R_S so that $|R_S| \leq 4|S|$, a translate of a $\frac{1}{2}$-fold dilate of R_S is contained in S, and such that a longest side of R_S has slope in the set $[0, 1]$.

Given a set $E \subset \mathbb{R}^2$, let $\pi_x(E) = \{x \in \mathbb{R} : (x, y) \in E \text{ for some } y \in \mathbb{R}\}$. Let $\pi_y(E) = \{y \in \mathbb{R} : (x, y) \in E \text{ for some } x \in \mathbb{R}\}$. Let $S \in \mathcal{B}$ be given. Let R_S be a rectangle associated to S as above. Associate to R_S a parallelogram P_S containing R_S whose left and right hand sides are parallel to the y-axis, such that the slope of the bottom side is greater than or equal to 0 and strictly less than 1, such that the length of the left hand side is 2^{-k} times the length of $\pi_x(P_S)$ for some nonnegative integer k, such that the length of $\pi_y(P_S)$ is an integer multiple of the length of the left hand side, such that $|P_S| \leq 32|R_S|$, and such that a translate of a $\frac{1}{32}$-fold dilate of P_S is contained in R_S. Figure 1 provides a depiction of a convex set S with its associated rectangle R_S and parallelogram P_S.

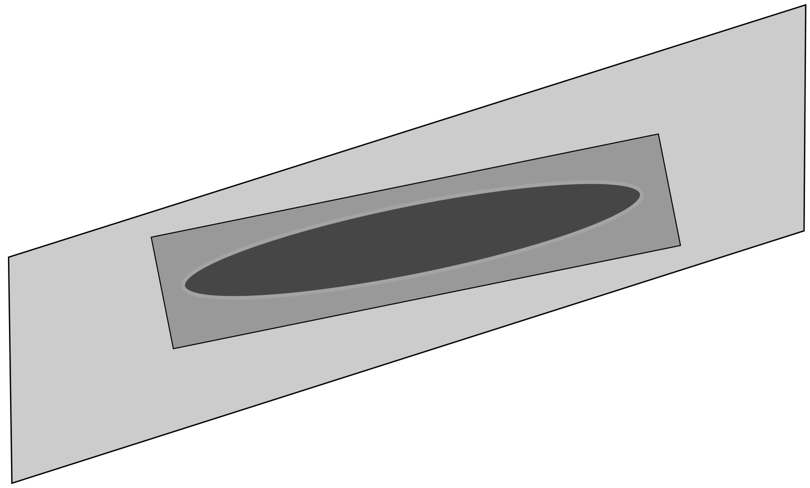

FIGURE 1. A Convex Set S with Associated Rectangle R_S and Parallelogram P_S

Let now $\tilde{\mathcal{P}}_{\mathcal{B}}$ consists of the set of all the homothecies of the parallelograms in $\{P_S : S \in \mathcal{B}\}$. For any measurable function f on $\mathbb{R}^2$ we have

$$\frac{1}{4096} M_{\tilde{\mathcal{P}}_{\mathcal{B}}} f(x) \leq M_{\mathcal{B}} f(x) \leq 128 M_{\tilde{\mathcal{P}}_{\mathcal{B}}} f(x) \ .$$

Accordingly, we recognize that it suffices to show that either the maximal operator $M_{\tilde{\mathcal{P}}_{\mathcal{B}}}$ is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \leq \infty$ or that $M_{\tilde{\mathcal{P}}_{\mathcal{B}}}$ is unbounded on $L^p(\mathbb{R}^2)$ for all $1 \leq p < \infty$.

We denote by $\mathcal{P}_{\mathcal{B}}$ the set of parallelograms in $\tilde{\mathcal{P}}_{\mathcal{B}}$ whose vertices are at $(0,0)$, $(0, 2^{-k})$, $(1, j2^{-k})$, and $(1, (j+1)2^{-k})$, where $k \geq 0$ is an integer and j is an integer satisfying $0 \leq j \leq 2^k - 1$. We now associate to the collection $\mathcal{P}_{\mathcal{B}}$ a subset of the dyadic tree, following the ideas of Gauvan in [4]. In doing so we need to introduce relevant terminology and notation.

Let $\mathscr{B}_0 = \{0\}$, let $\mathscr{B}_1 = \{00, 01\}$, and let $\mathscr{B}_2 = \{000, 001, 010, 011\}$. Continuing in this way, we can define the sets $\mathscr{B}_n$ recursively by

$$\mathscr{B}_{k+1} = \{0a_1 \cdots a_{k+1} : 0a_1 \cdots a_k \in \mathscr{B}_k \text{ and } a_{k+1} \in \{0,1\}\} \ .$$

Let $\mathscr{B} = \cup_{n=0}^{\infty} \mathscr{B}_n$. Note $\mathscr{B}$ consists of all finite strings of 0's and 1's such that the first number in the string is 0 . We endow $\mathscr{B}$ with the structure of a tree by adding edges between any element $0a_1 \cdots a_k$ of $\mathscr{B}$ and each of the two elements $0a_1 \cdots a_k 0$ and $0a_1 \cdots a_k 1$.

If $v \in \mathscr{B}_k$, we say v is of *height* k and write $h(v) = k$.

Let $\mathscr{T}$ be a subtree of $\mathscr{B}$. A *ray* in $\mathscr{T}$ is a (possibly infinite) maximal sequence of connected vertices $p_1, p_2, \ldots$ in $\mathscr{T}$ such that there exists a nonnegative integer k so that $p_1 \in \mathscr{B}_k$, $p_2 \in \mathscr{B}_{k+1}$, ... It is maximal in the sense that if the ray R is a finite sequence of vertices $p_1, p_2, \ldots, p_N$, then there is no vertex in $\mathscr{T}$ of height $h(p_N) + 1$ that is connected to

p_N. If a ray in $\mathscr{B}$ exists that contains vertices u and v where $h(u) < h(v)$, we say u is an *ancestor* of v and that v is a *descendant* of u.

We say that a vertex $v \in \mathscr{T}$ *splits in* $\mathscr{T}$ if v has two descendants in $\mathscr{T}$ of height $v(h)+1$.

We denote the set of rays in $\mathscr{T}$ whose vertex of lowest height is v by $\mathfrak{R}_{\mathscr{T}}(v)$. If R is a ray in $\mathscr{T}$ that contains exactly k vertices that split in $\mathscr{T}$, we say that $\mathrm{split}_{\mathscr{T}}(R) = k$. If R contains infinitely many vertices that split in $\mathscr{T}$, we say $\mathrm{split}_{\mathscr{T}}(R) = \infty$. If v is a vertex in $\mathscr{T}$, we define

$$\mathrm{split}_{\mathscr{T}}(v) = \min_{R \in \mathfrak{R}_{\mathscr{T}}(v)} \mathrm{split}_{\mathscr{T}}(R) \ .$$

We set

$$\mathrm{split}(v, \mathscr{T}) = \sup_{\mathscr{S} \subset \mathscr{T}} \mathrm{split}_{\mathscr{S}}(v) \ ,$$

where the supremum is over all subtrees $\mathscr{S}$ of $\mathscr{T}$, and

$$\mathrm{split}(\mathscr{T}) = \sup_{v \in \mathscr{T}} \mathrm{split}(v, \mathscr{T}) \ .$$

A tree $\mathscr{T} \subset \mathscr{B}$ is said to be *lacunary of order 0* if $\mathscr{T}$ is a ray in $\mathscr{B}$. A tree $\mathscr{T} \subset \mathscr{B}$ is said to be *lacunary of order* N if all the vertices of $\mathscr{T}$ that split in $\mathscr{T}$ lie on a lacunary tree of order $N-1$.

We define the *truncation* $\mathscr{S}^k$ of the set $\mathscr{S} \subset \mathscr{B}$ to be the subset of $\mathscr{S}$ whose vertices consist of the vertices of $\mathscr{S}$ of height less than or equal to k.

We may associate to any subset $\mathscr{S} \subset \mathscr{B}$ the associated subgraph $[\mathscr{S}]$ of $\mathscr{B}$ that is the smallest subtree of $\mathscr{B}$ containing all the vertices of $\mathscr{S}$ *together with all of their ancestors in* $\mathscr{B}$. Moreover, we may associate to $\mathscr{S}$ its *extension* $\mathscr{S}^*$ which we define to be the induced subgraph of $\mathscr{B}$ consisting of all the vertices of $[\mathscr{S}]$ together with all vertices of the form $0a_1a_2a_3 \cdots a_k 0 \cdots 0$ where $0a_1a_2a_3 \ldots a_k \in [\mathscr{S}]$.

Given the collection of parallelograms $\mathcal{P}_{\mathcal{B}}$ let $\mathscr{S}_{\mathcal{B}}$ denote the subset of $\mathscr{B}$ consisting of 0 and vertices of the form $a_0a_1a_2 \ldots a_k$ (note here a_0 is always 0) where $\mathcal{P}_{\mathcal{B}}$ contains the parallelogram denoted by $P_{a_0a_1\cdots a_k}$ with vertices

$$(0,0), (0, 2^{-k}), (1, \sum_{j=0}^{k} a_j 2^{-j}), (1, \sum_{j=0}^{k} a_j 2^{-j} + 2^{-k}) \ .$$

Lemma 1. *Suppose, given $N > 0$, there exists $k > 0$ such that* $\mathrm{split}[\mathscr{S}_{\mathcal{B}}^k] = N$*. Then $M_{\mathcal{B}}$ is not bounded on $L^p(\mathbb{R}^2)$ for any $1 \leq p < \infty$.*

Proof. We assume without loss of generality that the unit square $[0,1] \times [0,1]$ lies in $\mathcal{P}_{\mathcal{B}}$. Fix $N > 0$ and let k be such that $\mathrm{split}[\mathscr{S}_{\mathcal{B}}^k] = N$. Assume without loss of generality that k is the minimum number satisfying this condition, and in particular that $\mathrm{split}[\mathscr{S}_{\mathcal{B}}^{k-1}] = N - 1$. Following the pruning technique of Bateman [p. 62 of [1]] we prune the tree $[\mathscr{S}_{\mathcal{B}}^k]$ to yield a tree $\mathscr{P}_{\mathcal{B}}^k$ containing $v_0 = 0$ with $\mathrm{split}(v_0, \mathscr{P}_{\mathcal{B}}^k) = N$ satisfying the condition that, for every $R \in \mathfrak{R}_{\mathscr{P}_{\mathcal{B}}^k}(v_0)$ and every $j = 1, \ldots, N$, R contains exactly one splitting vertex v_j such that $\mathrm{split}(v_j, \mathscr{P}_{\mathcal{B}}^k) = j$.

Let now $\tilde{\mathscr{P}}^k_{\mathcal{B}}$ be an induced subgraph of $\mathscr{B}$ consisting of all of the vertices in $\mathscr{P}^k_{\mathcal{B}}$ together with all vertices of the form $a_0 a_1 \cdots a_n 0 \cdots 0$ in $\cup_{j=0}^k \mathscr{B}_j$ where $a_0 a_1 \cdots a_n$ is a member of $\mathscr{P}^k_{\mathcal{B}}$ that has no descendant in $\mathscr{P}^k_{\mathcal{B}}$.

Let $\sigma : \mathscr{B}^k \to \tilde{\mathscr{P}}^k_{\mathcal{B}}$ be a *sticky map.* In particular, we have that $\sigma(u)$ is a descendant of $\sigma(v)$ in $\tilde{\mathscr{P}}^k_{\mathcal{B}}$ whenever u is a descendant of v in $\mathscr{B}^k$. We also suppose that $h(\sigma(v)) = h(v)$ for all $v \in \mathscr{B}^k$. Now, we can associate to the map σ the set $K_\sigma \subset \mathbb{R}^2$ defined by

$$K_\sigma = \bigcup_{v \in \mathscr{B}_k} E_{\sigma(v)} \ ,$$

where, if $v = b_0 b_1 \cdots b_k$ and if $P_{\sigma(v)} = P_{a_0 a_1 \cdots a_k}$ is as defined earlier, then $E_{\sigma(v)}$ is the parallelogram with vertices at

$$(0, \sum_{j=0}^k b_j 2^{-j}), (0, \sum_{j=0}^k b_j 2^{-j} + 2^{-k}), (2, \sum_{j=0}^k b_j 2^{-j} + 2\sum_{j=0}^k a_j 2^{-j}), (2, \sum_{j=0}^k b_j 2^{-j} + 2\sum_{j=0}^k a_j 2^{-j} + 2^{-k}) \ .$$

Now, Bateman observed that there exists such a sticky map σ so that

$$|K_\sigma \cap ([0,1] \times \mathbb{R})| \gtrsim \frac{\log N}{N}$$

and

$$|K_\sigma \cap ([1,2] \times \mathbb{R})| \lesssim \frac{1}{N} \ .$$

Let us abbreviate the set $K_\sigma \cap ([0,1] \times \mathbb{R})$ as K_1 and the set $K_\sigma \cap ([1,2] \times \mathbb{R})$ as K_2.

It turns out that $M_{\tilde{\mathcal{P}}_{\mathcal{B}}} \chi_{K_2}(x) \geq \frac{1}{4}$ for every $x \in K_1$. To see this, note that if $x \in K_1$ then $x \in \tau_{(0, j_x \cdot 2^{-k})} P_{a_0 \cdots a_k}$ for some $a_0 \cdots a_k$ in $\tilde{\mathscr{P}}^k_{\mathcal{B}}$, with $j_x \cdot 2^{-k} = \sum_{n=0}^k b_n 2^{-n}$ and $\sigma(b_0 \cdots b_k) = a_0 \cdots a_k$. Of course, the parallelogram $P_{a_0 \cdots a_k}$ might not lie in $\mathcal{P}_{\mathcal{B}}$ itself. However, let $a_0 \cdots a_l$ be the nearest ancestor in $\mathscr{P}^k_{\mathcal{B}}$ to the vertex $a_0 \cdots a_k$. (We do allow $a_0 \cdots a_l$ to be $a_0 \cdots a_k$ itself provided $a_0 \cdots a_k \in \mathscr{S}_{\mathcal{B}}$.) *Note that $a_0 \cdots a_k$ is the only descendant of $a_0 \cdots a_l$ in $\tilde{\mathscr{P}}^k_{\mathcal{B}}$ of height k.* To see this, recognize that either $a_0 \cdots a_k$ lies in the tree $\mathscr{P}^k_{\mathcal{B}}$ itself, or it is of the form $a_0 \cdots a_l 0 \cdots 0$ where $a_0 \cdots a_l$ is a leaf of the tree $\mathscr{P}^k_{\mathcal{B}}$ (i.e., a member of $\mathscr{P}^k_{\mathcal{B}}$ having no descendant in $\mathscr{P}^k_{\mathcal{B}}$), noting that all of the descendants in $\tilde{\mathscr{P}}^k_{\mathcal{B}}$ of a leaf $a_0 \cdots a_l$ in $\mathscr{P}^k_{\mathcal{B}}$ are of the form $a_0 \cdots a_l 0 \cdots 0$. One should also recognize that if $a_0 \cdots a_l$ is a leaf of $\mathscr{P}^k_{\mathcal{B}}$, then $a_0 \cdots a_l$ must lie in $\mathscr{S}_{\mathcal{B}}$ itself. It is at this stage that we take advantage of the fact that σ is a sticky map. Since σ is a sticky map, we must have that $\sigma(b_0 \cdots b_s) = a_0 \cdots a_s$ for $l \leq s \leq k$, and hence $\sigma(v) = a_0 \cdots a_k$ whenever v is a descendant of height k of $b_0 \cdots b_l$ in $\mathscr{B}_k$. This tells us that x is contained in a union of 2^{k-l} a.e. disjoint parallelograms whose union forms a parallelogram P_x contained in K_1 that is a vertical translate of $P_{a_0 \cdots a_l} \in \mathcal{P}_{\mathcal{B}}$ contained in K_1. Not only that, but x is contained in a vertical translate E_x of the associated parallelogram $E_{a_0 \cdots a_l}$ that is contained in K_σ. As E_x is contained in a translate of a two-fold dilate of

$P_x \in \tilde{\mathcal{P}}_{\mathcal{B}}$, we have that $M_{\tilde{\mathcal{P}}_{\mathcal{B}}}\chi_{K_2}(x) \geq \frac{1}{4}$. In particular, we have that

$$\left|\left\{x \in \mathbb{R}^2 : M_{\tilde{\mathcal{P}}_{\mathcal{B}}}\chi_{K_2}(x) \geq \frac{1}{4}\right\}\right| \geq |K_1| \gtrsim \log N\,|K_2|\ .$$

As $N > 0$ is arbitrary, the lemma holds.

□

Lemma 2. *Suppose that there exists $N > 0$ so that for every $k > 0$ we have* $\mathrm{split}[\mathscr{S}^k_{\mathcal{B}}] \leq N$. *Then $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \leq \infty$.*

Proof. If for every $k > 0$ we have $\mathrm{split}[\mathscr{S}^k_{\mathcal{B}}] \leq N$, then $\mathscr{S}^*_{\mathcal{B}}$ is $N+1$ lacunary. Let $\Omega_{\mathcal{B}^*}$ denote the set of all directions associated to the $N+1$ lacunary tree $\mathscr{S}^*_{\mathcal{B}}$. (This could formally be defined as the smallest closed set in [0,1] containing all values of the form $\sum_{j=0}^{k} a_j 2^{-j}$ where $a_0 \cdots a_k \in \mathscr{S}^*_{\mathcal{B}}$.) Denote the associated directional maximal operator by $M_{\Omega_{\mathcal{B}^*}}$. As $\Omega_{\mathcal{B}^*}$ may be covered by a finite number of $N+1$ lacunary sets, we have by Bateman's Theorem that the directional maximal operator $M_{\Omega_{\mathcal{B}^*}}$ is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \leq \infty$. As $M_{\tilde{\mathcal{P}}_{\mathcal{B}}} f(x) \leq M_{\Omega_{\mathcal{B}^*}} f(x)$, we have $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \leq \infty$. □

As, given $\mathcal{B}$ the hypotheses of one of Lemma 1 or Lemma 2 must hold, we have proven the desired theorem.

□

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