SHARP WEAK TYPE ESTIMATES FOR A FAMILY OF SORIA BASES

DMITRY DMITRISHIN, PAUL HAGELSTEIN, AND ALEX STOKOLOS

ABSTRACT. Let $\mathcal{B}$ be a collection of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and such that $\mathcal{B}$ contains parallelepipeds with side lengths of the form $s, \frac{2^N}{s}, t$, where $s, t > 0$ and N lies in a nonempty subset S of the natural numbers. We show that if S is an infinite set, then the associated geometric maximal operator $M_{\mathcal{B}}$ satisfies the weak type estimate

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}} f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right)^2$$

but does not satisfy an estimate of the form

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}} f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \phi\left(\frac{|f|}{\alpha}\right)$$

for any convex increasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying the condition

$$\lim_{x \to \infty} \frac{\phi(x)}{x(\log(1 + x))^2} = 0 \, .$$

1. INTRODUCTION

This paper is concerned with sharp weak type estimates for a class of maximal operators naturally arising from work surrounding the so-called Zygmund conjecture in multiparameter harmonic analysis. Let us recall that the *strong maximal operator* M is defined on $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$M f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f| \, ,$$

where the supremeum is over all rectangular parallelepipeds in $\mathbb{R}^n$ containing x whose sides are parallel to the coordinate axes. An important inequality associated to the strong maximal operator is

$$|\{x \in \mathbb{R}^n : M f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right)^{n-1} \, .$$

This inequality may be found in de Guzmán [5,6] (see also the related paper [3] of A. Córdoba and R. Fefferman as well as the paper [1] of Capri and Fava) and may be used to provide

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a proof of the classical *Jessen-Marcinkiewicz-Zygmund Theorem* [8], which tells us that the integral of any function in $L(\log^+ L)^{n-1}(\mathbb{R}^n)$ is strongly differentiable.

Now, the strong maximal operator in $\mathbb{R}^n$ is associated to an n-parameter basis of rectangular parallelepipeds. It is natural to consider weak type estimates for maximal operators in $\mathbb{R}^n$ associated to k-parameter bases. The *Zygmund Conjecture* in this regard is the following:

Conjecture 1 (Zygmund Conjecture; now disproven)**.** *Let $\mathcal{B}$ be a collection of rectangular parallelepipeds in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes and whose sidelengths are of the form*

$$\phi_1(t_1, \ldots, t_k), \ldots, \phi_n(t_1, \ldots, t_k)$$

where the functions ϕ_i are nonnegative and increasing in each variable separately. Define the associated maximal operator $M_{\mathcal{B}}$ by

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| \, .$$

Then $M_{\mathcal{B}}$ satisfies the weak type estimate

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha} \left(1 + \log^+ \frac{|f|}{\alpha}\right)^{k-1} \, . \tag{1.1}$$

This conjecture was disproven by Soria in [9]. That being said, it does hold in many important cases. For example, A. Córdoba proved in [2] that the Zygmund Conjecture holds in the case that $\mathcal{B}$ consists of rectangular parallelepipeds in $\mathbb{R}^3$ with sides parallel to the coordinate axes and whose sidelengths are of the form $s, t, \phi(s,t)$, where ϕ is nonnegative and increasing in the variables s, t separately. Of particular interest to us in this paper is the following extension of Córdoba's result due to Soria in [9]:

Proposition 1. *Let $\mathcal{B}$ be a collection of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes. Furthermore, suppose that, given a parallelepiped R in $\mathcal{B}$ of sidelengths r_1, r_2, r_3 and another parallelepiped R' in $\mathcal{B}$ of sidelengths r'_1, r'_2, r'_3, if $r_1 > r'_1$, then either $r_2 > r'_2$ or $r_3 > r'_3$. Then*

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}} f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha} \left(1 + \log^+ \frac{|f|}{\alpha}\right) \, .$$

Note that this proposition encompasses bases that can be quite different in character than the ones consider by Córdoba. In particular, in [9] Soria mentions as an example the basis of parallelepipeds with sidelengths of the form $s, t, \frac{1}{t}$.

At this point we introduce another strand of research associated to Zygmund's Conjecture. It is natural to consider, given a translation invariant basis $\mathcal{B}$ of rectangular parallelepipeds, whether or not the *sharp* weak type estimate associated to $M_{\mathcal{B}}$ must be of the form

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha} \left(1 + \log^+ \frac{|f|}{\alpha}\right)^{k-1}$$

for some *integer* $1 \leq k \leq n$. In [10], Stokolos proved the following:

Proposition 2. *Let $\mathcal{B}$ be a translation invariant basis of rectangles in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes. If $\mathcal{B}$ does not satisfy the weak type $(1,1)$ estimate*

$$|\{x \in \mathbb{R}^2 : M_{\mathcal{B}}f(x) > \alpha\}| \leq C \int_{\mathbb{R}^2} \frac{|f|}{\alpha}$$

then $M_{\mathcal{B}}$ satisfies the weak type estimate

$$\left|\left\{x \in \mathbb{R}^2 : M_{\mathcal{B}}f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^2} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right)$$

but does not *satisfy a weak type estimate of the form*

$$|\{x \in \mathbb{R}^2 : M_{\mathcal{B}}f(x) > \alpha\}| \leq C \int_{\mathbb{R}^2} \phi\left(\frac{|f|}{\alpha}\right)$$

for any nonnegative convex increasing function ϕ such that $\phi(x) = o(x \log x)$ as x tends to infinity.

In essence, this proposition tells us that, if $\mathcal{B}$ is a translation invariant basis of rectangles in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes, then the optimal weak type estimate for $M_{\mathcal{B}}$ must be inequality 1.1 for $k = 1$ or $k = 2$. Optimal weak type estimates of this form when, say, $k = \frac{3}{2}$ are ruled out. The proof of Stokolos' result is very delicate and involves the idea of *crystallization* that we will return to.

It is of interest that Proposition 2 has at the present time never been extended to encompass translation invariant bases consisting of (some, but not all) rectangular parallelepipeds in dimensions 3 or higher. In particular, one might expect that the optimal weak type estimate for the maximal operator associated to such a basis of parallelepipeds in $\mathbb{R}^3$ would be inequality 1.1 when $n = 3$ and k is either 1, 2, or 3.

The purpose of this paper is, motivated by Propositions 1 and 2 above, to consider sharp weak type estimates associated to the translation invariant basis of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and whose sidelengths are of the form $s, \frac{2^N}{s}, t$, where $s, t > 0$ and N lies in a nonempty subset S of the natural numbers. The end result, although not its proof, is strikingly straightforward and is stated as follows:

Theorem 1. *Let $\mathcal{B}$ be a collection of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and such that $\mathcal{B}$ consists of all parallelepipeds with side lengths of the form $s, \frac{2^N}{s}, t$, where $s, t > 0$ and N lies in a nonempty subset S of the natural numbers.*

If S is a finite set, then the associated geometric maximal operator $M_{\mathcal{B}}$ satisfies the weak type estimate

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}}f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right) \tag{1.2}$$

but does not satisfy an estimate of the form

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}}f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \phi\left(\frac{|f|}{\alpha}\right)$$

for any convex increasing function $\phi : [0,\infty) \to [0,\infty)$ *satisfying the condition*

$$\lim_{x\to\infty} \frac{\phi(x)}{x(\log(1+x))} = 0 \ .$$

If S is an infinite set, then the associated geometric maximal operator $M_{\mathcal{B}}$ satisfies the weak type estimate

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}}f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right)^2$$

but does not satisfy an estimate of the form

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}}f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \phi\left(\frac{|f|}{\alpha}\right)$$

for any convex increasing function $\phi : [0,\infty) \to [0,\infty)$ *satisfying the condition*

$$\lim_{x\to\infty} \frac{\phi(x)}{x(\log(1+x))^2} = 0 \ .$$

The remainder of the paper is devoted to a proof of this theorem. Note that for inequality 1.2, it is easily seen that the constant C is at most linearly dependent on the number of elements in S, although the sharp dependence of C on the number of elements of S is potentially a quite difficult issue that we do not treat here. The primary content of the above theorem is the sharpness of the weak type estimate of $M_{\mathcal{B}}$ in the case that S is infinite. In harmonic analysis we typically show that an optimal weak type estimate on a maximal operator is sharp by testing the operator on a bump function or the characteristic function of a small interval or rectangular parallelepiped. This can be done, for instance, with the Hardy-Littlewood maximal operator, the strong maximal operator, or even the maximal operator associated to rectangles whose sides are parallel to the axes with sidelengths of the form $t, \frac{1}{t}$ [9]. However, in dealing with maximal operators associated to rare bases of the type featured in Theorem 1, such simple functions *do not* provide examples illustrating the sharpness of the optimal weak type results, and more delicate constructions such as will be seen here are needed.

We remark that a recent paper of D'Aniello and Moonens [4] also treats the subject of translation invariant rare bases; in particular they provide sufficient conditions on a rare basis $\mathcal{B}$ for the estimate 1.1 to be sharp when $k = n$. However, certain bases covered in Theorem 1 (such as when $S = \{2^{m^m} : m \in \mathbb{N}\}$) do not fall into the scope of those considered in their paper, although the interested reader is strongly encouraged to consult it.

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2. Crystallization and Preliminary Weak Type Estimates

In this section, we shall introduce a collection of two-dimensional "crystals" that we will use to prove Theorem 1. We remark that similar types of crystalline structures were used by Stokolos in [10, 11, 12] as well as by Hagelstein and Stokolos in [7].

Let $m_1 < m_2 < \cdots$ be an increasing sequence of natural numbers. We may associate to this sequence and any $k \in \mathbb{N}$ a set in $[0, 2^{m_k}]$ denoted by $Y_{\{m_j\}_{j=1}^k}$ defined by

$$Y_{\{m_j\}_{j=1}^k} = \left\{ t \in [0, 2^{m_k}] : \sum_{j=1}^{k} r_0\left(\frac{t}{2^{m_j}}\right) = k \right\} .$$

Here $r_0(t)$ denotes the standard Rademacher function defined on $[0, 1)$ by

$$r_0(t) = \chi_{[0,\frac{1}{2}]}(t) - \chi_{(\frac{1}{2},1)}(t)$$

and extended to be 1-periodic on $\mathbb{R}$.

Note that

$$\mu_1(Y_{\{m_j\}_{j=1}^k}) = 2^{-k} 2^{m_k} .$$

Associated to the set $Y_{\{m_j\}_{j=1}^k}$ is the *crystal* $Q_{\{m_j\}_{j=1}^k} \subset [0, 2^{m_k}] \times [0, 2^{m_k}]$ defined by

$$Q_{\{m_j\}_{j=1}^k} = Y_{\{m_j\}_{j=1}^k} \times Y_{\{m_j\}_{j=1}^k} .$$

Note

$$\mu_2(Q_{\{m_j\}_{j=1}^k}) = 2^{-2k} 2^{2m_k} .$$

Here μ_j refers to the Lebesgue measure on $\mathbb{R}^j$.

We also associate to $\{m_j\}_{j=1}^k$ the geometric maximal operator $M_{\{m_j\}_{j=1}^k}$ defined on $L^1_{loc}(\mathbb{R}^2)$ by

$$M_{\{m_j\}_{j=1}^k} f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f| ,$$

where the supremum is over all rectangles in $\mathbb{R}^2$ containing x whose sides are parallel to the coordinate axes with areas in the set $\{2^{m_1}, \ldots, 2^{m_k}\}$.

In the case that the context is clear, we may refer to the set $Y_{\{m_j\}_{j=1}^k}$ simply as Y_k, the set $Q_{\{m_j\}_{j=1}^k}$ simply as Q_k, and the maximal operator $M_{\{m_j\}_{j=1}^k}$ simply as M_k.

A few basic observations regarding the sets Y_k and Q_k are in order.

First, note that Y_{k+1} is a disjoint union of $\frac{2^{m_{k+1}-1}}{2^{m_k}}$ copies of Y_k. In fact, defining the translation $\tau_s E$ of a set E in $\mathbb{R}$ by $\chi_{\tau_s E}(x) = \chi_E(x-s)$, we have

$$Y_{k+1} = \bigcup_{l=0}^{\frac{2^{m_{k+1}-1}}{2^{m_k}}-1} \tau_{l2^{m_k}} Y_k \ .$$

Furthermore, by induction we see that if $1 \leq r \leq k$ we have Y_{k+1} is a disjoint union of

$$\frac{2^{m_{k+1}-1}}{2^{m_k}} \cdot \frac{2^{m_k-1}}{2^{m_{k-1}}} \cdots \frac{2^{m_{r+1}-1}}{2^{m_r}} = 2^{m_{k+1}-m_r-k+r-1}$$

copies of Y_r , with

$$Y_{k+1} = \bigcup_{\substack{(l_r,\ldots,l_k) \\ 0 \leq l_i \leq 2^{m_{i+1}-m_i-1}-1}} \tau_{l_r 2^{m_r}} \tau_{l_{r+1} 2^{m_{r+1}}} \cdots \tau_{l_k 2^{m_k}} Y_r \ .$$

We also remark that the average of χ_{Y_k} over $[0, 2^{m_j}]$ is exactly 2^{-j}, and moreover the average of χ_{Y_k} over any translate $\tau_{l_j 2^{m_j}} \tau_{l_{j+1} 2^{m_{j+1}}} \cdots \tau_{l_{k-1} 2^{m_{k-1}}} [0, 2^{m_j}]$ with $0 \leq l_i \leq 2^{m_{i+1}-m_i-1} - 1$ is also 2^{-j} . Observe that the number of such translates is

$$2^{m_{j+1}-m_j-1} \cdot 2^{m_{j+2}-m_{j+1}-1} \cdots 2^{m_k-m_{k-1}-1} = 2^{m_k-m_j+j-k}.$$

We now consider how M_k acts on χ_{Q_k}. We will do so in the special case that, for $1 \leq j \leq \frac{k}{2}$ we have that $m_{k-j} \leq m_{k-j+1} - m_j$. (This will be the case if the m_j increase rapidly in j, for example if $m_{j+1} \geq 2m_j$ for all j.)

Fix now $1 \leq j \leq \frac{k}{4}$. We are going to show that there exist

$$2^{m_k-m_{k-j+1}+m_j-j} \cdot 2^{m_k-m_j-k+j} = 2^{2m_k-m_{k-j+1}-k}$$

pairwise a.e. disjoint rectangles with sides parallel to the coordinate axes in $[0, 2^{m_k}] \times [0, 2^{m_k}]$ whose areas are all $2^{m_{k-j+1}}$ and such that the average of χ_{Q_k} over each of these rectangles is 2^{-k}. Moreover, each of these rectangles will be a translate of $[0, 2^{m_j}] \times [0, 2^{m_{k-j+1}-m_j}]$. Accordingly, the measure of the union of these rectangles will be 2^{2m_k-k}.

We have already indicated above that the average of χ_{Y_k} over each of $2^{m_k-m_j-k+j}$ pairwise a.e. disjoint translates of $[0, 2^{m_j}]$ is 2^{-j}. Somewhat more technically, we now need to prove that the average of χ_{Y_k} over $2^{m_k-m_{k-j+1}+m_j-j}$ pairwise a.e. disjoint intervals of length $2^{m_{k-j+1}-m_j}$ is equal to 2^{j-k}.

Note that the average of χ_{Y_k} over $[0, 2^{m_{k-j}}]$ is 2^{j-k} as well as any translate $\tau[0, 2^{m_{k-j}}]$ of this interval where τ is of the form $l \cdot 2^{m_{k-j}}$ for $0 \leq l \leq 2^{m_{k-j+1}-m_j-m_{k-j}} - 1$. The union of these intervals is the interval $I := [0, 2^{m_{k-j+1}-m_j}]$ over which the average of χ_{Y_k} is 2^{j-k}. It is especially important to recognize here that

$$Y_k \cap [0, 2^{m_{k-j+1}-1}] = \bigcup_{i=0}^{2^{m_{k-j+1}-m_{k-j}-1}} \tau_{i2^{m_{k-j}}} Y_{k-j} \ ,$$

where the latter is a pairwise a.e. disjoint union. It is here that we need the condition that $m_{k-j} \leq m_{k-j+1} - m_j$, so that $[0, 2^{m_{k-j+1}-m_j}]$ can be tiled by pairwise a.e. disjoint intervals of length $2^{m_{k-j}}$ over which the average of $\chi_{Y_{k-j}}$ is 2^{j-k}.

Now, $[0, 2^{m_k}]$ contains many pairwise a.e. disjoint translates of $I \cap Y_k$, each of whom being contained in a collection of translates of I that are themselves pairwise a.e. disjoint; we count them here. The number of translates is the number of pairwise a.e. disjoint translates of I whose union is the left half of $[0, 2^{m_{k-j+1}}]$ (which is $2^{m_{k-j+1}-1-m_{k-j+1}+m_j} = 2^{m_j-1}$) times the number of translates of Y_{k-j+1} needed to form Y_k (which is $2^{m_k - m_{k-j+1} - k + (k-j+1)} = 2^{m_k - m_{k-j+1} - j + 1}$.) Hence the total number of translates is

$$2^{m_j - 1} \cdot 2^{m_k - m_{k-j+1} - j + 1} = 2^{m_j + m_k - m_{k-j+1} - j} .$$

Hence, Y_k contains $2^{m_j + m_k - m_{k-j+1} - j}$ pairwise a.e. disjoint intervals of length $2^{m_{k-j+1} - m_j}$ over each of which the average of χ_{Y_k} is 2^{j-k}. As we have already shown that the average of χ_{Y_k} over each of $2^{m_k - m_j - k + j}$ pairwise a.e. disjoint translates of $[0, 2^{m_j}]$ is 2^{-j}, we have then that there exist $2^{m_j + m_k - m_{k-j+1} - j} \cdot 2^{m_k - m_j - k + j} = 2^{2m_k - m_{k-j+1} - k}$ pairwise a.e. disjoint rectangles in $[0, 2^{m_k}] \times [0, 2^{m_k}]$ of size $2^{m_{k-j+1} - m_j} \cdot 2^{m_j} = 2^{m_{k-j+1}}$ over each of which the average of χ_{Q_k} is $2^{-j} \cdot 2^{j-k} = 2^{-k}$. Note the measure of the union of these rectangles is

$$2^{2m_k - m_{k-j+1} - k} \cdot 2^{m_{k-j+1}} = 2^{2m_k - k} .$$

We come now to a crucial observation. By the construction of Y_k, any dyadic interval of length 2^{m_j} is at most only half filled by the translates of intervals of length $2^{m_{j-1}}$ such that the union of those translates acting on Y_{j-1} is Y_j . Accordingly, the union of the above $2^{2m_k - m_{k-j+1} - k}$ pairwise a.e. disjoint rectangles in $[0, 2^{m_k}] \times [0, 2^{m_k}]$ of size $2^{m_{k-j+1}}$ over each of which the average of χ_{Q_k} is 2^{-k} is at most only half filled by the corresponding set of rectangles of size $2^{m_{k-(j-1)+1}}$. Hence the union of all the rectangles R in $[0, 2^{m_k}] \times [0, 2^{m_k}]$ whose sides are parallel to the coordinate axes and of area in the set $\{2^{m_{k-j}} : j = 1, \ldots, \lceil \frac{k}{4} \rceil\}$ and such that the average of χ_{Q_k} over R is greater than or equal to 2^{-k} must exceed $\frac{1}{2} \cdot \frac{k}{4} \cdot 2^{2m_k - k} = \frac{k}{8} 2^{2m_k - k}$.

This series of observations leads to the proof of the following lemma.

Lemma 1. *Let the geometric maximal operator $M_{\{m_j\}_{j=1}^k}$ and the set $Q_{\{m_j\}_{j=1}^k}$ be defined as above. Suppose for $1 \leq j \leq \frac{k}{2}$ we have that $m_{k-j} \leq m_{k-j+1} - m_j$. Then*

$$\mu_2 \left(\left\{ x \in [0, 2^{m_k}] \times [0, 2^{m_k}] : M_{\{m_j\}_{j=1}^k} \chi_{Q_{\{m_j\}_{j=1}^k}} (x) \geq 2^{-k} \right\} \right) \geq \frac{k}{8} 2^{2m_k - k} = \frac{1}{8} \frac{k}{2^{-k}} \mu_2 \left(Q_{\{m_j\}_{j=1}^k} \right) .$$

3. Proof of Theorem 1

Proof of Theorem 1. Let $\mathcal{B}$ be a collection of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and such that $\mathcal{B}$ contains parallelepipeds with side lengths of the form $s, \frac{2^N}{s}, t$, where $t > 0$ and S is a nonempty set consisting of natural numbers.

If S is a finite set, then the associated geometric maximal operator $M_{\mathcal{B}}$ is comparable to the maximal operator averaging over rectangular parallelepipeds with side lengths of the form $s, \frac{1}{s}, t$. In [9], Soria showed that this operator maps $L(1+\log^+ L)(\mathbb{R}^3)$ continuously into weak $L^1(\mathbb{R}^3)$ but does not map any larger Orlicz class into weak $L^1(\mathbb{R}^3)$. So Theorem 1 holds in this case.

Suppose now S is an infinite set. Note that the maximal operator $M_{\mathcal{B}}$ is dominated by the strong maximal operator in $\mathbb{R}^3$, so the weak type estimate

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}} f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right)^2$$

automatically holds.

Since S is an infinite set, there exists a subset $\{m_j\}_{j=1}^{\infty}$ of S satisfying the condition that $2m_j \leq m_{j+1}$ for all j. So the hypothesis of Lemma 1 holds for $\{m_j\}_{j=1}^{k}$ for all k.

For each natural number k, we let $Z_k \subset [0, 2^{m_k}] \times [0, 2^{m_k}] \times [0, 2^k]$ be defined by

$$Z_k = Q_k \times [0,1] \ .$$

To show the estimate

$$\left|\left\{x \in \mathbb{R}^3 : M_{\mathcal{B}} f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^3} \phi\left(\frac{|f|}{\alpha}\right)$$

does *not* hold for any convex increasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying the condition

$$\lim_{x\to\infty} \frac{\phi(x)}{x(\log(1+x))^2} = 0 \ ,$$

it suffices to show that

$$\mu_3\left(\left\{x \in [0, m_k] \times [0, m_k] \times [0, 2^k] : M_{\mathcal{B}} \chi_{Z_k}(x) \geq 2^{-k}\right\}\right) \geq \frac{1}{32} \frac{k^2}{2^{-k}} \mu_3(Z_k) \ .$$

Fix $1 \leq r \leq k$. Note that, just as Y_k is a disjoint union of $2^{m_k - m_r - k + r}$ copies of Y_r , we have that Q_k is a disjoint union of $2^{2(m_k - m_r - k + r)}$ copies of Q_r, with each of these copies being contained in pairwise a.e. disjoint squares of sidelength 2^{m_r}. By Lemma 1, for each one of these squares $\tilde{Q}$,

$$\mu_2\left(\left\{x \in \tilde{Q} : M_r \chi_{\tilde{Q}\cap Q_k}(x) \geq 2^{-r}\right\}\right) \geq \frac{r}{8} 2^{2m_r - r} \ .$$

Note each of the rectangles associated to the maximal operator M_r has sidelength in the set $\{2^{m_1}, \ldots, 2^{m_r}\} \subset \{2^{m_1}, \ldots, 2^{m_k}\}$ and hence for any of these rectangles R the associated parallelepiped $R \times [0, 2^{k-r}]$ lies in the basis $\mathcal{B}$. Note that if

$$\frac{1}{\mu_2(R)} \int_R \chi_{\tilde{Q}\cap Q_k} \geq 2^{-r} \ ,$$

then

$$\frac{1}{\mu_3(R\times[0,2^{k-r}])}\int_{R\times[0,2^{k-r}]}\chi_{Q_k\times[0,1]} \geq 2^{-r}2^{r-k} = 2^{-k}\ .$$

Taking into account only the top half of these parallelepipeds, for any one of the above squares $\tilde{Q}$ we obtain

$$\mu_3\left(\left\{x\in[0,2^{m_k}]\times[0,2^{m_k}]\times[2^{k-r-1},2^{k-r}] : M_{\mathcal{B}}\chi_{Z_k}(x)\geq 2^{-k}\right\}\right)\geq$$

$$2^{2(m_k-m_r-k+r)}\mu_2\left(\left\{x\in\tilde{Q} : M_r\chi_{\tilde{Q}\cap Q_k}(x)\geq 2^{-r}\right\}\right)\cdot 2^{k-r-1}$$

$$\geq 2^{2(m_k-m_r-k+r)}\frac{r}{8}2^{2m_r-r}\cdot 2^{k-r-1} \;=\; \frac{r}{16}2^{2m_k-k}\ .$$

We now take advantage of the fact that, for different values of r, the sets $[0,2^{m_k}]\times[0,2^{m_k}]\times[2^{k-r-1},2^{k-r}]$ are pairwise a.e. disjoint. In particular, we have

$$\begin{aligned}
&\mu_3\left(\left\{x\in[0,2^{m_k}]\times[0,2^{m_k}]\times[0,2^{k}] : M_{\mathcal{B}}\chi_{Z_k}(x)\geq 2^{-k}\right\}\right)\\
&\geq\sum_{r=1}^{k}\mu_3\left(\left\{x\in[0,2^{m_k}]\times[0,2^{m_k}]\times[2^{k-r-1},2^{k-r}] : M_{\mathcal{B}}\chi_{Z_k}(x)\geq 2^{-k}\right\}\right)\\
&\geq\sum_{r=1}^{k}\frac{r}{16}2^{2m_k-k}\geq\frac{1}{32}\frac{k^2}{2^k}2^{2m_k}=\frac{1}{32}\frac{k^2}{2^{-k}}\mu_3(Z_k)\ ,
\end{aligned}$$

as desired.

□

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D. D.: Department of Applied Mathematics, Odessa National Polytechnic University, Odessa 65044, Ukraine

Email address: dmitrishin@opu.ua

P. H.: Department of Mathematics, Baylor University, Waco, Texas 76798

Email address: paul_hagelstein@baylor.edu

A. S.: Department of Mathematical Sciences, Georgia Southern University, Statesboro, Georgia 30460

Email address: astokolos@GeorgiaSouthern.edu