CÓRDOBA-FEFFERMAN COLLECTIONS IN HARMONIC ANALYSIS

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ABSTRACT. Córdoba-Fefferman collections are defined and used to characterize functions whose corresponding maximal functions are locally integrable. Córdoba-Fefferman collections are used to show that, if M_x and M_y denote the one-dimensional Hardy-Littlewood maximal operators in the horizontal and vertical directions in $\mathbf{R}^2$, M_{HL} denotes the standard Hardy-Littlewood maximal operator in $\mathbf{R}^2$, and f is a measurable function supported in the unit square $Q = [0,1] \times [0,1]$, then $\int_Q M_{HL} f \sim \int_Q M_x f + \int_Q M_y f$.

We begin by introducing the following definition:

Definition 1. *Let β be a countable collection of Lebesgue measurable subsets of the unit n-cube $I^{(n)}$ in $\mathbf{R}^n$ of positive measure. A (possibly finite) subset $\{R_i\}$ of β is said to be a Córdoba-Fefferman collection with respect to β (denoted by $\{R_i\} \in CFC(\beta)$) if and only if there exists an enumeration $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \ldots$ of the elements of $\{R_i\}$ such that $|\tilde{R}_i \cap \left(\cup_{j<i} \tilde{R}_j\right)| \leq \frac{1}{2}|\tilde{R}_i|$ for each $i = 2, 3, 4, \ldots$.*

In [1] A. Córdoba and R. Fefferman used what we are now calling Córdoba-Fefferman collections to characterize geometric maximal operators that are of weak type (p,p) for $p > 1$. The purpose of this paper is to show that Córdoba-Fefferman collections may also be used to estimate the integrals of maximal functions. The primary result in this regard is the following.

Theorem 2. *Let β be a countable collection of Lebesgue measurable subsets of the unit n-cube $I^{(n)}$ in $\mathbf{R}^n$ of positive measure. Let β be such that for any point x in $I^{(n)}$, $x \in R$ for some $R \in \beta$. Define the maximal operator M_β on $L^1\left(I^{(n)}\right)$ by*

$$M_\beta f(x) = \sup_{x \in R \in \beta} \frac{1}{|R|} \int_R |f(y)|\, dy. \tag{1}$$

Suppose M_β satisfies the (Tauberian) condition

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$$\left|\left\{x \in I^{(n)} : M_\beta \chi_E(x) \geq \frac{1}{2}\right\}\right| \leq C_\beta |E| \tag{2}$$

for all measurable sets $E \subset I^{(n)}$. *Then if* $f \in L^1\left(I^{(n)}\right)$,

$$\int_{I^{(n)}} M_\beta f \sim \sup_{\{R_i\} \in CFC(\beta)} \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i}. \tag{3}$$

In particular,

$$\frac{1}{2} \sup_{\{R_i\} \in CFC(\beta)} \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i} \leq \int_{I^{(n)}} M_\beta f \leq 4\, C_\beta \sup_{\{R_i\} \in CFC(\beta)} \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i}.$$

Proof. We assume without loss of generality that $f \in L^\infty\left(I^{(n)}\right)$, $f \not\equiv 0$. We begin by showing

$$\int_{I^{(n)}} M_\beta f \leq 4\, C_\beta \sup_{\{R_i\} \in CFC(\beta)} \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i}.$$

Let $\epsilon > 0$. It suffices to show there exists $\{R_i\} \in CFC(\beta)$ such that

$$4\, C_\beta \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i} \geq \int_{I^{(n)}} M_\beta f \ - \ \epsilon.$$

Let m be a positive integer such that

$$0 \leq \int_{\left\{x \in I^{(n)} : M_\beta f(x) > 2^m\right\}} M_\beta f < \frac{\epsilon}{3}.$$

Let $R_1, R_2, R_3, \ldots$ be an enumeration of the elements of β.
Let $R_{1,1}$ be the first element on the list of the R_i such that

$$\frac{1}{|R_{1,1}|} \int_{R_{1,1}} |f| > 2^m.$$

Assuming $R_{1,1}, R_{1,2}, \ldots, R_{1,k}$ have been chosen, let $R_{1,k+1}$ be the first element on the list of the R_i such that

$$\left|R_{1,k+1} \cap \left(\bigcup_{i=1}^{k} R_{1,i}\right)\right| \leq \frac{1}{2} |R_{1,k+1}|$$

and

$$\frac{1}{|R_{1,k+1}|} \int_{R_{1,k+1}} |f| > 2^m.$$

(If such an element of β does not exist, we stop the selection procedure at this point.)

In this manner, a (possibly finite) sequence $R_{1,1}, R_{1,2}, \ldots$ is attained.

Let j_1 be an integer such that

$$\left|\bigcup_{i=1}^{j_1} R_{1,i}\right| > \frac{1}{2}\left|\bigcup_{i=1}^{\infty} R_{1,i}\right|.$$

We now renumerate β (allowing for multiple counting of individual elements) as

$$(*_2) \qquad R_{1,1}, R_{1,2}, \ldots, R_{1,j_1}, R_1, R_2, R_3, \ldots.$$

Now let $R_{2,1} = R_{1,1}$. Let $R_{2,2}$ be the first element on the list $(*_2)$ such that $|R_{2,2} \cap R_{2,1}| \leq \frac{1}{2}|R_{2,2}|$ and

$$\frac{1}{|R_{2,2}|}\int_{R_{2,2}} |f| > 2^{m-1}.$$

Assuming $R_{2,1}, R_{2,2}, \ldots, R_{2,k}$ have been selected, let $R_{2,k+1}$ be the first element on the list $(*_2)$ such that

$$\left|R_{2,k+1} \cap \left(\bigcup_{i=1}^{k} R_{2,i}\right)\right| \leq \frac{1}{2}|R_{2,k+1}|$$

and

$$\frac{1}{|R_{2,k+1}|}\int_{R_{2,k+1}} |f| > 2^{m-1}.$$

In this manner the sequence $R_{2,1}, R_{2,2}, \ldots$ is generated.

Let $j_2 \geq j_1$ be an integer such that

$$\left|\bigcup_{i=1}^{j_2} R_{2,i}\right| > \frac{1}{2}\left|\bigcup_{i=1}^{\infty} R_{2,i}\right|.$$

Note that $R_{1,1} = R_{2,1}, R_{1,2} = R_{2,2}, \ldots, R_{1,j_1} = R_{2,j_1}$.

We continue inductively. Assume that $R_{n,1}, R_{n,2}, \ldots, R_{n,j_n}$ have been selected. We renumerate β as

$$(*_{n+1}) \qquad R_{n,1}, R_{n,2}, \ldots, R_{n,j_n}, R_1, R_2, R_3, \ldots.$$

Let $R_{n+1,1} = R_{n,1}$. Let $R_{n+1,2}$ be the first element on the list $(*_{n+1})$ such that $|R_{n+1,2} \cap R_{n+1,1}| \leq \frac{1}{2}|R_{n+1,2}|$ and

$$\frac{1}{|R_{n+1,2}|}\int_{R_{n+1,2}} |f| > 2^{m-n}.$$

Assuming $R_{n+1,1},\ldots,R_{n+1,k}$ have been selected, let $R_{n+1,k+1}$ be the first element on the list $(*_{n+1})$ such that

$$\left|R_{n+1,k+1}\cap\left(\bigcup_{i=1}^{k}R_{n+1,i}\right)\right|\leq\frac{1}{2}\left|R_{n+1,k+1}\right|$$

and

$$\frac{1}{\left|R_{n+1,k+1}\right|}\int_{R_{n+1,k+1}}|f|>2^{m-n}.$$

In this manner, a sequence $R_{n+1,1},R_{n+1,2},\ldots$ is selected. Let $j_{n+1}\geq j_n$ be an integer such that

$$\left|\bigcup_{i=1}^{j_{n+1}}R_{n+1,i}\right|>\frac{1}{2}\left|\bigcup_{i=1}^{\infty}R_{n+1,i}\right|.$$

Note that $R_{n,1}=R_{n+1,1}\ ,\ R_{n,2}=R_{n+1,2}\ ,\ \ldots,R_{n,j_n}=R_{n+1,j_n}$. This is clear, as $R_{n,1},R_{n,2},\ldots,R_{n,j_n}$ are the first j_n elements of β chosen in the procedure for selecting the $R_{n+1,i}$.

We now relate $\left|\bigcup_{i=1}^{j_{n+1}}R_{n+1,i}\right|$ to $\left|\left\{x\in I^{(n)}:M_{\beta}f\left(x\right)>2^{m-n}\right\}\right|$. Suppose for some $p\in I^{(n)}$ that $M_{\beta}f\left(p\right)>2^{m-n}$. Then for some $R\in\beta$, $\frac{1}{|R|}\int_R|f|>2^{m-n}$. Then $|R\cap(\cup_{i=1}^{\infty}R_{n+1,i})|\geq\frac{1}{2}|R|$. Hence

$$M_{\beta}\left(\chi_{\bigcup_{i=1}^{\infty}R_{n+1,i}}\right)\left(p\right)\geq\frac{1}{2}.$$

As

$$\left|\left\{x\in I^{(n)}:M_{\beta}\chi_E\left(x\right)\geq\frac{1}{2}\right\}\right|\leq C_{\beta}\left|E\right|$$

for all measurable sets $E\subset I^{(n)}$, we see that

$$\left|\left\{x\in I^{(n)}:M_{\beta}f\left(x\right)>2^{m-n}\right\}\right|\leq C_{\beta}\left|\bigcup_{i=1}^{\infty}R_{n+1,i}\right|\leq 2\,C_{\beta}\left|\bigcup_{i=1}^{j_{n+1}}R_{n+1,i}\right|. \tag{4}$$

We now let l be a positive integer such that $2^{m-l}<\frac{\epsilon}{3}$. Then

$$\left|\int_{\left\{x\in I^{(n)}:2^{m-l}<M_{\beta}f(x)<2^{m}\right\}}M_{\beta}f\ -\ \int_{I^{(n)}}M_{\beta}f\right|<\frac{2\epsilon}{3}.$$

We now compare $\int_{\left\{x\in I^{(n)}:2^{m-l}<M_{\beta}f(x)<2^{m}\right\}}M_{\beta}f$ to $\int_{I^{(n)}}|f|\cdot\sum_{i=1}^{j_{l+1}}\chi_{R_{l+1,i}}$. Let

$$\lambda\left(\alpha\right)=\left|\left\{x\in I^{(n)}:M_{\beta}f\left(x\right)>\alpha\right\}\right|,$$

$$\mu(\alpha) = \left| \left\{ x \in I^{(n)} : |f(x)| \cdot \sum_{i=1}^{j_{l+1}} \chi_{R_{l+1,i}}(x) > \alpha \right\} \right|,$$

and

$$\omega(\alpha) = \left| \left\{ x \in I^{(n)} : \sum_{i=1}^{j_{l+1}} \left(\frac{1}{|R_{l+1,i}|} \int_{R_{l+1,i}} |f| \right) \cdot \chi_{R_{l+1,i}}(x) > \alpha \right\} \right|.$$

Suppose $2^{m-l} \leq \alpha \leq 2^m$. Let r be the largest integer such that $2^r \leq \alpha$. Clearly $m-l \leq r \leq m$. Now

$$R_{l+1,1} = R_{m-r+1,1},\ R_{l+1,2} = R_{m-r+1,2},\ \ldots, R_{l+1,j_{m-r+1}} = R_{m-r+1,\, j_{m-r+1}}.$$

Also, by (4) we have

$$\lambda(2^r) \leq 2\, C_\beta \left| \bigcup_{i=1}^{j_{m-r+1}} R_{m-r+1,i} \right|.$$

As

$$\frac{1}{|R_{m-r+1,i}|} \int_{R_{m-r+1,i}} |f| > 2^r \ \text{ for } i = 1, 2, \ldots, j_{m-r+1},$$

we see that $\lambda(2^r) \leq 2\, C_\beta\, \omega(2^r)$. Hence $\lambda(\alpha) \leq 2\, C_\beta\, \omega\left(\frac{\alpha}{2}\right)$ for $2^{m-l} \leq \alpha \leq 2^m$. So

$$\begin{aligned}
\int_{\left\{x \in I^{(n)} : 2^{m-l} < M_\beta f(x) < 2^m\right\}} M_\beta f &\leq \int_{2^{m-l}}^{2^m} \lambda(\alpha)\, d\alpha \ + \ \frac{\epsilon}{3} \\
&\leq 2\, C_\beta \int_{2^{m-l}}^{2^m} \omega\left(\frac{\alpha}{2}\right) d\alpha \ + \ \frac{\epsilon}{3} \\
&\leq 4\, C_\beta \int_0^\infty \omega(\alpha)\, d\alpha \ + \ \frac{\epsilon}{3} \\
&= 4\, C_\beta \int_0^\infty \mu(\alpha)\, d\alpha \ + \ \frac{\epsilon}{3} \\
&= 4\, C_\beta \int_{I^{(n)}} |f| \cdot \sum_{i=1}^{j_{l+1}} \chi_{R_{l+1,i}} \ + \ \frac{\epsilon}{3}\,.
\end{aligned}$$

Hence

$$\int_{I^{(n)}} M_\beta f \leq 4\, C_\beta \int_{I^{(n)}} |f| \cdot \sum_{i=1}^{j_{l+1}} \chi_{R_{l+1,i}} + \epsilon\,.$$

As ϵ is an arbitrary positive real number and $\{R_{\ell+1,i}\} \in CFC(\beta)$, we see that

$$\int_{I^{(n)}} M_\beta f \leq 4\, C_\beta \sup_{\{R_i\} \in CFC(\beta)} \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i}\,,$$

as desired.

We now show

$$\int_{I^{(n)}} M_\beta f \geq \frac{1}{2} \sup_{\{R_i\} \in CFC(\beta)} \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i}.$$

Let $\{R_i\} \in CFC(\beta)$. Without loss of generality, we assume

$$\left| R_i \cap \left(\bigcup_{j=1}^{i-1} R_j \right) \right| \leq \frac{1}{2} |R_i| \text{ for } i = 2, 3, \ldots.$$

It suffices to show that

$$\int_{I^{(n)}} M_\beta f \geq \frac{1}{2} \int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i}.$$

Let $E_1 = R_1$ and $E_k = R_k \backslash \left(\cup_{i=1}^{k-1} R_i \right)$ for $k = 2, 3, \ldots$. Let

$$Tf(x) = \sum_k \left(\frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \chi_{E_k}(x).$$

Clearly $Tf(x) \leq M_\beta f(x)$. Also,

$$\begin{aligned}
\int_{I^{(n)}} Tf &= \int_{I^{(n)}} \sum_k \left(\frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \chi_{E_k}(x) \, dx \\
&= \sum_k \int_{I^{(n)}} \left(\frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \cdot \chi_{E_k}(x) \, dx \\
&\geq \frac{1}{2} \sum_k \int_{I^{(n)}} \left(\frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \cdot \chi_{R_k}(x) \, dx \\
&\qquad \left(\text{since } |E_k| \geq \frac{1}{2} |R_k| \right) \\
&= \frac{1}{2} \sum_k \int_{R_k} |f(y)| \, dy \\
&= \frac{1}{2} \int_{I^{(n)}} |f| \cdot \sum_k \chi_{R_k}.
\end{aligned}$$

So

$$\int_{I^{(n)}} |f| \cdot \sum_i \chi_{R_i} \leq 2 \int_{I^{(n)}} M_\beta f \, ,$$

as desired. □

To illustrate the role of the Tauberian condition in the above theorem, we consider the following example:

For $0 < \delta < \frac{1}{10}$, define β_δ by

$$\beta_\delta = \left\{ A \subset [0,1] : A = [0,\delta] \cup \left[x, x+\delta^2\right] \text{ for some } x \in \left[0, 1-\delta^2\right] \right\}.$$

Note that $C_{\beta_\delta} \gtrsim \delta^{-1}$, since

$$M_{\beta_\delta}\left(\chi_{[0,\delta]}\right)(x) > \frac{1}{2} \text{ for all } x \in [0,1].$$

If $\{R_i\} \in CFC(\beta_\delta)$, then $\{R_i\}$ has only one element, say $R_1 = [0,\delta] \cup [x, x+\delta^2]$ for some $x \in [0, 1-\delta^2]$. So

$$\int_0^1 \chi_{[0,1]} \cdot \chi_{R_1} \leq 2\delta.$$

However,

$$\int_0^1 M_{\beta_\delta} \chi_{[0,1]} = 1.$$

Note that although the ratio of

$$\int_0^1 M_{\beta_\delta} \chi_{[0,1]} \quad \text{to} \quad \sup_{\{R_i\} \in CFC(\beta_\delta)} \int_Q \chi_{[0,1]} \cdot \sum_i \chi_{R_i}$$

may be arbitrarily large (depending on the value of δ), the ratio is still bounded by C_{β_δ}.

Before indicating applications of the above theorem, we list some basic definitions.

Definition 3 (Hardy-Littlewood maximal function)**.** *Let f be a measurable function defined on $\mathbf{R}^n$. Let $B(p,r)$ denote the Euclidean ball in $\mathbf{R}^n$ centered at p of radius r, and let $|B(p,r)|$ denote the Lebesgue measure of $B(p,r)$. The Hardy-Littlewood maximal function of f is defined on $\mathbf{R}^n$ by*

$$M_{HL} f(p) = \sup_{r>0} \frac{1}{|B(p,r)|} \int_{B(p,r)} |f(z)|\, dz. \tag{5}$$

Definition 4 (Strong maximal function)**.** *Let f be a measurable function defined on $\mathbf{R}^2$. The strong maximal function of f is defined on $\mathbf{R}^2$ by*

$$M_S f(x,y) = \sup_{\substack{x_1<x<x_2 \\ y_1<y<y_2}} \frac{1}{(x_2-x_1)(y_2-y_1)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} |f(u,v)|\, dv\, du. \tag{6}$$

Definition 5 (Horizontal maximal function)**.** *Let f be a measurable function defined on $\mathbf{R}^2$. The horizontal maximal function of f is defined on $\mathbf{R}^2$ by*

$$M_x f(u,v) = \sup_{u_1<u<u_2} \frac{1}{u_2-u_1} \int_{u_1}^{u_2} |f(w,v)|\, dw. \tag{7}$$

Definition 6 (Vertical maximal function)**.** *Let f be a measurable function defined on $\mathbf{R}^2$. The vertical maximal function of f is defined on $\mathbf{R}^2$ by*

$$M_y f(u,v) = \sup_{v_1<v<v_2} \frac{1}{v_2-v_1} \int_{v_1}^{v_2} |f(u,w)|\, dw. \tag{8}$$

In the following the unit square $I^{(2)}$ in $\mathbf{R}^2$ will be denoted by Q. Also, if a given maximal operator M_β is naturally associated to a collection β, as, say, M_S is associated to the set of rectangles with sides parallel to the axes, we will frequently denote the Córdoba-Fefferman collection $CFC(\beta)$ by $CFC(M_\beta)$.

We now turn to one of the most useful applications of Córdoba-Fefferman collection theory. Suppose we are given the maximal operators M_α, M_β, and M_γ, all of which satisfy the desired Tauberian condition. Suppose also we want to show that, for some measurable function f supported on Q, $\int_Q M_\alpha f \lesssim \int_Q M_\beta f + \int_Q M_\gamma f$. One strategy for doing this would be to show that, given an arbitrary collection $\{A_i\} \in CFC(M_\alpha)$, one can produce $\{B_i\} \in CFC(M_\beta)$ and $\{C_i\} \in CFC(M_\gamma)$ such that

$$\sum_i \chi_{A_i}(p) \leq \sum_i \chi_{B_i}(p) + \sum_i \chi_{C_i}(p)$$

for almost every p in Q. Theorem 2 would then yield the desired result. The primary difficulty in applying this strategy is that the production of the $\{B_i\} \in CFC(M_\beta)$ and $\{C_i\} \in CFC(M_\gamma)$ can be a quite complicated matter, and in some situations literally impossible. However, in many cases one can modify this strategy by using the geometry of f to choose a particular $\{A_i\} \in CFC(M_\alpha)$ in such a manner that not only $\int_Q M_\alpha f \sim \int_Q |f| \cdot \sum_i \chi_{A_i}$, but also the production of the $\{B_i\} \in CFC(M_\beta)$ and $\{C_i\} \in CFC(M_\gamma)$ can follow in a geometrically intuitive fashion. Actually proving that $\int_Q M_\alpha f \sim \int_Q |f| \cdot \sum_i \chi_{A_i}$ often requires a duplication of large parts of the proof of Theorem 2 in the special case determined by the geometry of f and the desired properties of the collection $\{A_i\}$. We illustrate these ideas in the proof of the following.

Lemma 7. *Let f be a nonnegative measurable function supported on Q such that $f(x_1,y_1) \geq f(x_2,y_2)$ whenever $0 \leq x_1 \leq x_2$, $0 \leq y_1 \leq y_2$. Then*

$$\int_Q M_{HL} f \leq C \left(\int_Q M_x f + \int_Q M_y f \right).$$

for some universal constant C.

Proof. As M_{HL} and the operators M_x, M_y are bounded on $L^2(Q)$, we may assume without loss of generality that f is smooth. Hence without loss of

generality we may assume $f \in L^{\infty}(Q)$. Let m be the largest integer such that $2^m \leq \|f\|_{L^{\infty}(Q)}$. Let

$$E\left(2^{m-j+1}\right) = \left\{x \in Q : M_{HL} f(x) > 2^{m-j+1}\right\}.$$

For each positive integer j and to each $p \in E\left(2^{m-j+1}\right)$ associate a square $Q_{p,m-j+1}$ containing p such that

$$\frac{1}{|Q_{p,m-j+1}|} \int_{Q_{p,m-j+1}} f > 2^{m-j+1}$$

and such that one of the edges of $Q_{p,m-j+1}$ is contained in one of the coordinate axes. Note that such a square exists, since f is nonincreasing in each variable separately. Associate now to $Q_{p,m-j+1}$ a dyadic subsquare $Q'_{p,m-j+1}$ contained in $Q_{p,m-j+1}$ which has an edge contained in one of the coordinate axes, such that $\left|Q'_{p,m-j+1}\right| \geq \frac{1}{16}\left|Q_{p,m-j+1}\right|$, and such that no dyadic subsquare of $Q_{p,m-j+1}$ of the same size contains a point closer to the origin than any point of $Q'_{p,m-j+1}$.

If $Q'_{p,m-j+1}$ intersects the origin, let $Q''_{p,m-j+1}$ be $Q'_{p,m-j+1}$. Otherwise, let $Q''_{p,m-j+1}$ be the dyadic square with the same area as $Q'_{p,m-j+1}$ such that $Q''_{p,m-j+1}$ and $Q'_{p,m-j+1}$ share an edge, $Q''_{p,m-j+1}$ has an edge contained in one of the coordinate axes, and such that $Q''_{p,m-j+1}$ contains a point q which is closer to the origin than any point of $Q'_{p,m-j+1}$. Note that

$$\frac{1}{\left|Q''_{p,m-j+1}\right|} \int_{Q''_{p,m-j+1}} f > 2^{m-j+1}. \tag{9}$$

Let $16Q''_{p,m-j+1}$ be the square concentric to $Q''_{p,m-j+1}$ whose sides are parallel to the axes and whose volume is 16^2 times that of $Q''_{p,m-j+1}$. Note that $Q_{p,m-j+1} \subset 16Q''_{p,m-j+1}$.

For each positive integer j, let $Q_{1,m-j+1}$, $Q_{2,m-j+1}$, ... enumerate the squares in $\left\{Q''_{p,m-j+1} : p \in E\left(2^{m-j+1}\right)\right\}$ that are not properly contained in any of the other squares in $\left\{Q''_{p,m-j+1} : p \in E\left(2^{m-j+1}\right)\right\}$. Note that since the $Q_{i,m-j+1}$ are dyadic, the squares $Q_{1,m-j+1}$, $Q_{2,m-j+1}$, ... are disjoint. Also

$$\left|\bigcup_{i=1}^{\infty} Q_{i,m-j+1}\right| \geq 2^{-8}\left|E\left(2^{m-j+1}\right)\right|, \tag{10}$$

since $E\left(2^{m-j+1}\right) \subset \bigcup_{i=1}^{\infty} 16Q_{i,m-j+1}$. Also, by (9), each $Q_{i,m-j+1}$ satisfies

$$\frac{1}{|Q_{i,m-j+1}|} \int_{Q_{i,m-j+1}} f > 2^{m-j+1}. \tag{11}$$

For each positive integer j, let ρ_{m-j+1} be a positive integer such that

$$\left| \bigcup_{i=1}^{\rho_{m-j+1}} Q_{i,m-j+1} \right| \geq \frac{1}{2} \left| \bigcup_{i=1}^{\infty} Q_{i,m-j+1} \right|. \tag{12}$$

We now form the following sequence of dyadic squares: let $Q_1 = Q_{1,m}, Q_2 = Q_{2,m}, \ldots, Q_{\rho_m} = Q_{\rho_m,m}$, $Q_{\rho_m+1} = Q_{1,m-1}, Q_{\rho_m+2} = Q_{2,m-1}$, $\ldots$,$Q_{\rho_m+\rho_{m-1}} = Q_{\rho_{m-1},m-1}$, $Q_{\rho_m+\rho_{m-1}+1} = Q_{1,m-2}, \ldots$.

Let now $\tilde{Q}_1 = Q_1$. Let $\tilde{Q}_2$ be the first Q on the list $Q_1, Q_2, Q_3, \ldots$ such that $\left| Q \cap \tilde{Q}_1 \right| \leq \frac{1}{2} |Q|$. Assuming $\tilde{Q}_1, \ldots, \tilde{Q}_k$ have been chosen, let $\tilde{Q}_{k+1}$ be the first Q on the list $Q_1, Q_2, Q_3, \ldots$ such that $\left| Q \cap \left(\bigcup_{i=1}^{k} \tilde{Q}_i \right) \right| \leq \frac{1}{2} |Q|$. In this manner the sequence $\left\{ \tilde{Q}_i \right\}$ is generated.

Now, let $j_1 \leq j_2 \leq j_3 \leq \ldots$ be such that, if $Q \in \left\{ \tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_{j_k} \right\}$, then $\frac{1}{|Q|} \int_Q |f| > 2^{m-k+1}$, and

$$\left| \bigcup_{i=1}^{j_k} \tilde{Q}_i \right| \geq \frac{1}{4} \left| \bigcup_{i=1}^{\infty} Q_{i,m-k+1} \right|.$$

This is possible via (12) and the selection rule for the $\tilde{Q}_i$.

Note that (10) implies

$$\left| \bigcup_{i=1}^{j_k} \tilde{Q}_i \right| \geq \frac{1}{4} \cdot 2^{-8} \left| E\left(2^{m-k+1}\right) \right|. \tag{13}$$

Pick $\epsilon > 0$. Let ℓ be an integer greater than 1 such that $2^{m-\ell+1} < \epsilon$. Then

$$\left| \int_{\left\{ x \in Q : 2^{m-\ell+1} < M_{HL} f(x) < 2^{m+1} \right\}} M_{HL} f - \int_Q M_{HL} f \right| < \epsilon.$$

We now compare $\int_{\left\{ x \in Q : 2^{m-\ell+1} < M_{HL} f(x) < 2^{m+1} \right\}} M_{HL} f$ to $\int_Q f \cdot \sum_{i=1}^{j_\ell} \chi_{\tilde{Q}_i}$.

Let

$$\lambda(\alpha) = \left| \{ x \in Q : M_{HL} f(x) > \alpha \} \right|,$$

$$\mu(\alpha) = \left| \left\{ x \in Q : f(x) \cdot \sum_{i=1}^{j_\ell} \chi_{\tilde{Q}_i}(x) > \alpha \right\} \right|,$$

and

$$\omega(\alpha) = \left| \left\{ x \in Q : \sum_{i=1}^{j_\ell} \left(\frac{1}{\left| \tilde{Q}_i \right|} \int_{\tilde{Q}_i} f \right) \cdot \chi_{\tilde{Q}_i}(x) > \alpha \right\} \right|.$$

Now, suppose $2^{m-\ell+1} \leq \alpha < 2^{m+1}$. Let r be the largest integer such that $2^r \leq \alpha$. Hence $m - \ell + 1 \leq r \leq m$. By (13) and the remarks preceding (13) we see that $\lambda(2^r) \leq 4 \cdot 2^8 \omega(2^r)$. Hence $\lambda(\alpha) \leq 2^{10} \omega\left(\frac{\alpha}{2}\right)$ for $2^{m-\ell+1} \leq \alpha < 2^{m+1}$. So

$$\int_{\left\{x \in Q : 2^{m-\ell+1} < M_{HL} f(x) < 2^{m+1}\right\}} M_{HL} f \leq \int_{2^{m-\ell+1}}^{2^{m+1}} \lambda(\alpha)\, d\alpha + \epsilon$$

$$\begin{aligned}
&\leq 2^{10} \int_{2^{m-\ell+1}}^{2^{m+1}} \omega\left(\frac{\alpha}{2}\right) d\alpha + \epsilon \\
&\leq 2^{11} \int_0^{\infty} \omega(\alpha)\, d\alpha + \epsilon \\
&= 2^{11} \int_0^{\infty} \mu(\alpha)\, d\alpha + \epsilon \\
&= 2^{11} \int_Q f \cdot \left(\sum_{i=1}^{j_\ell} \chi_{\tilde{Q}_i} \right) + \epsilon.
\end{aligned}$$

Hence

$$\int_Q M_{HL} f \leq 2^{11} \int_Q f \cdot \left(\sum_{i=1}^{j_\ell} \chi_{\tilde{Q}_i} \right) + 2\epsilon. \tag{14}$$

For $i = 1, 2$ we generate a finite sequence $\left\{\tilde{Q}_{i,j}\right\}$ as follows: Let $\tilde{Q}_{i,1}$ be the first square Q on the list $\tilde{Q}_1$, $\tilde{Q}_2$, …, $\tilde{Q}_{j_\ell}$ which contains an element whose i'th component is 0 (i.e. there is an element $p = (p_1, p_2) \in Q$ with $p_i = 0$.) For each positive integer j, let $\tilde{Q}_{i,j}$ be the j'th square on the list with this property (if such a square exists). Since each of the $\tilde{Q}_i$ intersect one of the coordinate axes, each cube $\tilde{Q}_i$ will be an element in at least one of the two sequences. Suppose now that for each $i = 1, 2$ the sequence $\left\{\tilde{Q}_{i,j}\right\}$ has q_i squares. Then by (14),

$$\int_Q M_{HL} f \leq 2^{11} \int_Q f \cdot \left(\sum_{i=1}^{q_1} \chi_{\tilde{Q}_{1,i}} + \sum_{i=1}^{q_2} \chi_{\tilde{Q}_{2,i}} \right) + 2\epsilon. \tag{15}$$

For $i = 1, 2$ and each positive integer j, define the collection $\mathcal{R}_{i,j}$ by $\mathcal{R}_{i,j} = \{R \subseteq Q : R$ is a rectangle whose sides are parallel to the axes, one of the edges of R with smallest length is parallel to the line $x_i = 0$, and one of the edges of R with smallest length has length 2^{-j} $\}$.

(For example, an element of $\mathcal{R}_{1,3}$ would be the rectangle with corners at the points $\left(0, \frac{1}{8}\right)$, $\left(0, \frac{1}{4}\right)$, $\left(\frac{1}{2}, \frac{1}{8}\right)$, and $\left(\frac{1}{2}, \frac{1}{4}\right)$.)

Define the maximal operators $M_{i,j}$ by

$$M_{i,j}f(x) = \sup_{x \in R \in \mathcal{R}_{i,j}} \frac{1}{|R|} \int_R |f(y)|\, dy. \tag{16}$$

Since $f \in C^\infty(Q)$ it is clear that

$$\lim_{j\to\infty} \int_Q M_{1,j}f = \int_Q M_x f.$$

and

$$\lim_{j\to\infty} \int_Q M_{2,j}f = \int_Q M_y f.$$

For our added convenience, we will frequently denote the operator M_x by M_1 and M_y by M_2. Hence the above equalities may be expressed as

$$\lim_{j\to\infty} \int_Q M_{i,j}f = \int_Q M_i f.$$

It follows that there exists an integer, designated by j_0, such that

$$\left| \int_Q M_{i,j_0}f - \int_Q M_i f \right| < \frac{\epsilon}{2^{13}} \tag{17}$$

for $i = 1, 2$.

We assume without loss of generality that

$$2^{-j_0} \leq \inf \left\{ \left|\tilde{Q}_1\right|^{\frac{1}{2}}, \left|\tilde{Q}_2\right|^{\frac{1}{2}}, \ldots, \left|\tilde{Q}_{j_\ell}\right|^{\frac{1}{2}} \right\}.$$

We now show that for each $i = 1, 2$,

$$\int_Q f \cdot \sum_{j=1}^{q_i} \chi_{\tilde{Q}_{i,j}} \leq 2 \int_Q M_{i,j_0} f.$$

Let $\tilde{R}_{i,j,1}, \ldots, \tilde{R}_{i,j,2^{j_0}\cdot\left|\tilde{Q}_{i,j}\right|^{\frac{1}{2}}}$ be the $2^{j_0} \cdot \left|\tilde{Q}_{i,j}\right|^{\frac{1}{2}}$ disjoint rectangles in $\mathcal{R}_{i,j_0}$ of equal area whose union is $\tilde{Q}_{i,j}$. Let $\gamma_{ij} = 2^{j_0} \cdot \left|\tilde{Q}_{i,j}\right|^{\frac{1}{2}}$.

For $i = 1, 2$, let $\tilde{R}_{i,1} = \tilde{R}_{i,1,1}$, $\tilde{R}_{i,2} = \tilde{R}_{i,1,2}$, $\ldots$, $\tilde{R}_{i,\gamma_{i1}} = \tilde{R}_{i,1,\gamma_{i1}}$, $\tilde{R}_{i,\gamma_{i1}+1} = \tilde{R}_{i,2,1}$, $\ldots$, $\tilde{R}_{i,\gamma_{i1}+\gamma_{i2}} = \tilde{R}_{i,2,\gamma_{i2}}$, $\tilde{R}_{i,\gamma_{i1}+\gamma_{i2}+1} = \tilde{R}_{i,3,1}$, $\ldots$, $\tilde{R}_{i,\gamma_{i1}+\ldots+\gamma_{iq_i}} = \tilde{R}_{i,q_i,\gamma_{iq_i}}$.

Since for $i = 1, 2$ the squares $\tilde{Q}_{i,j}$ for $j = 1, \ldots, q_i$ are all dyadic and intersect the line $\{x_i = 0\}$, the selection rule for the $\tilde{Q}_k$ yield that

$$\left| \tilde{R}_{i,j} \cap \left(\bigcup_{k=1}^{j-1} \tilde{R}_{i,k} \right) \right| \leq \frac{1}{2} \left| \tilde{R}_{i,j} \right| \tag{18}$$

for each $j = 2, 3, \ldots, \gamma_{i1} + \ldots + \gamma_{iq_i}$.

Let $\gamma_i = \gamma_{i1} + \ldots + \gamma_{iq_i}$. Now, (18) implies that $\{R_{i,1}, R_{i,2}, \ldots, R_{i,\gamma_i}\} \in CFC\left(\mathcal{R}_{i,j_0}\right)$. Hence Theorem 2 tells us that

$$\int_Q |f| \cdot \sum_{j=1}^{\gamma_i} \chi_{\tilde{R}_{i,j}} \leq 2 \int_Q M_{i,j_0} f. \tag{19}$$

Now by the construction of the $\tilde{R}_{i,j}$ it is clear that

$$\sum_{j=1}^{\gamma_i} \chi_{\tilde{R}_{i,j}} = \sum_{j=1}^{q_i} \chi_{\tilde{Q}_{i,j}}. \tag{20}$$

(19) and (20) then yield

$$\int_Q |f| \cdot \sum_{j=1}^{q_i} \chi_{\tilde{Q}_{i,j}} \leq 2 \int_Q M_{i,j_0} f. \tag{21}$$

(15), (17), and (21) yield

$$\int_Q M_{HL} f \leq 2^{8+4} \left(\int_Q M_1 f + \int_Q M_2 f \right) + 3\epsilon. \tag{22}$$

As ϵ is arbitrarily small, we see that

$$\int_Q M_{HL} f \leq C \left(\int_Q M_1 f + \int_Q M_2 f \right) \tag{23}$$

for some universal constant C, completing the proof. □

The preceding lemma and the following rearrangement result will enable us to prove that if f is a measurable function supported on Q, then $\int_Q M_{HL} f \sim \int_Q M_x f + \int_Q M_y f$.

Lemma 8. *Let f be a nonnegative measurable function supported on Q. Let $\tilde{f}$ be the function supported on Q which is nonincreasing in x (i. e. $\tilde{f}(x_1, y) \geq \tilde{f}(x_2, y)$ whenever $0 \leq x_1 \leq x_2 \leq 1$, $0 \leq y \leq 1$) and such that, for each $y \in [0,1]$, $\tilde{f}(\cdot, y)$ and $f(\cdot, y)$ are equidistributed. Then*

$$\int_Q M_y \tilde{f} \leq c \int_Q M_y f,$$

where c is a universal constant.

Proof. Let $\alpha > 0$. Let $\lambda(\alpha) = |\{(u,v) \in Q : M_y f(u,v) > \alpha\}|$. Define $\tilde{\lambda}(\alpha)$ similarly. It suffices to show that $\tilde{\lambda}(\alpha) \leq 400 \cdot \lambda\left(\frac{\alpha}{64}\right)$.

Without loss of generality, assume f is smooth on Q. Take the Calderón-Zygmund decomposition of f with respect to α on each vertical segment in

$\{s \times [0,1], s \in [0,1]\}$ of Q, yielding for each $x \in [0,1]$ disjoint sets $Q_{x,j,\alpha} \subseteq [0,1]$ such that $\alpha < \frac{1}{|Q_{x,j,\alpha}|} \int_{Q_{x,j,\alpha}} f(x,z)\,dz \leq 2\alpha$. (In the case that $\int_0^1 f(x,z)\,dz > 2\alpha$, set $Q_{x,1,\alpha} = [0,1]$.) Note that $f(p) \leq \alpha$ for almost every p in the complement of $\cup_{\substack{x \in [0,1] \\ j \in \mathbf{Z}_+}} (x \times Q_{x,j,\alpha})$. For $\tilde{f}$ one may produce the associated sets $\tilde{Q}_{x,j,\alpha}$ in a similar fashion.

Let $E_\alpha = \left\{ (x,y) \in Q : y \in \cup_{\substack{x \in [0,1] \\ j \in \mathbf{Z}_+}} Q_{x,j,\alpha} \right\}$. Define $\tilde{E}_\alpha$ similarly. It suffices to show that $\left|\tilde{E}_{4\alpha}\right| \leq 2\,|E_\alpha|$. To see this, recall from the theory of the Hardy-Littlewood maximal operator M_{HL} that if g is a measurable function supported on $[0,1]$, $g \geq 0$, $\int_0^1 g \leq 2\alpha$, $\lambda_{HL}(\alpha) = |\{x \in [0,1] : M_{HL} g(x) > \alpha\}|$, and $E_{HL,\alpha} = \cup Q_{j,\alpha}$, where the $Q_{j,\alpha}$ are the intervals obtained by taking the Calderón-Zygmund decomposition of g with respect to α, then $|E_{HL,\alpha}| \leq \lambda_{HL}\left(\frac{\alpha}{2}\right) \leq 200\left|E_{HL,\frac{\alpha}{8}}\right|$. This readily yields that, if $\left|\tilde{E}_{4\alpha}\right| \leq 2\,|E_\alpha|$, then $\tilde{\lambda}(\alpha) \leq 200\left|\tilde{E}_{\frac{\alpha}{8}}\right| \leq 400\left|E_{\frac{\alpha}{32}}\right| \leq 400\ \lambda\left(\frac{\alpha}{64}\right)$. Hence $\tilde{\lambda}(\alpha) \leq 400 \cdot \lambda\left(\frac{\alpha}{64}\right)$, as desired.

To show that $\left|\tilde{E}_{4\alpha}\right| \leq 2\,|E_\alpha|$ we proceed as follows:

First we consider the special case in which $\int_0^1 f(x,y)\,dy \leq \alpha$ for any $x \in [0,1]$. Having taken the above described Calderón-Zygmund decomposition of f with respect to α, we yield the disjoint sets $Q_{x,j,\alpha} \subset [0,1]$ for each $x \in [0,1]$ and the associated set E_α. Now, for almost every p in the complement of E_α, $f(p) \leq \alpha$. So if S is a measurable subset of Q and $|S| > 2\,|E_\alpha|$, then $\int_S f \leq 2\alpha\,|S|$. Now let $\phi : Q \to Q$ be a measure-preserving bijection such that $\tilde{f}(\phi(p)) = f(p)$ for any $p \in Q$. Using ϕ we see that $\left|\tilde{E}_{4\alpha}\right| \leq 2\,|E_\alpha|$. Otherwise, if $\left|\tilde{E}_{4\alpha}\right| > 2\,|E_\alpha|$ we would have $\frac{1}{\left|\tilde{E}_{4\alpha}\right|} \int_{\tilde{E}_{4\alpha}} \tilde{f} = \frac{1}{\left|\tilde{E}_{4\alpha}\right|} \int_{\phi^{-1}\left(\tilde{E}_{4\alpha}\right)} f \leq 2\alpha$ by the above. But $\frac{1}{\left|\tilde{E}_{4\alpha}\right|} \int_{\tilde{E}_{4\alpha}} \tilde{f} > 4\alpha$ by the construction of $\tilde{E}_{4\alpha}$. So $\left|\tilde{E}_{4\alpha}\right| \leq 2\,|E_\alpha|$ if $\int_0^1 f(x,y)\,dy \leq \alpha$ for all $x \in [0,1]$.

Now we let f be an arbitrary nonnegative smooth function on Q. Without loss of generality assume there exists $c \in (0,1)$ such that $\int_0^1 f(x,y)\,dy > \alpha$ if $x < c$, and $\int_0^1 f(x,y)\,dy \leq \alpha$ if $x \geq c$. Calderón-Zygmund decompose f with respect to α as before, yielding the $Q_{x,j,\alpha}$ and E_α. Note if $x < c$, $Q_{x,1,\alpha} = [0,1]$.

For each $y \in [0,1]$ we define the functions $f_y(x)$ on $[0,1]$ by $f_y(x) = f(x,y)$. We construct a function $f_y'(x)$ on $[0,1]$ equidistributed to $f_y(x)$ such that $f_y'(x_2) \leq f_y'(x_1)$ if $x_2 \geq c \geq x_1$ and $f_y'(x) \leq f_y(x)$ if $x \geq c$ as follows:

Let $B_y = \left\{x \in [0,c) : f_y(x) < \tilde{f}_y(c)\right\}$. Let $A_y \subset \left\{x \in [c,1] : f_y(x) \geq \tilde{f}_y(c)\right\}$ be such that the measure of its interior is equal to $|B_y|$. Let A_y^o and B_y^o denote the interiors of A_y and B_y, respectively. Let $\phi_y : A_y^o \to B_y^o$ be a measure-preserving bijection such that if $x \in A_y^o$, $\left|\left\{b \in B_y^o : b < \phi_y(x)\right\}\right| = \left|\left\{a \in A_y^o : a < x\right\}\right|$. Define $f_y'(x)$ by

$$f_y'(x) = \begin{cases} f_y(x) & x \notin A_y^o \cup B_y^o \\ f_y\left(\phi_y^{-1}(x)\right) & x \in B_y^o \\ f_y(\phi_y(x)) & x \in A_y^o. \end{cases}$$

Note if $x > c$, $f_y'(x) \leq f_y(x)$. We define the function f' on Q by $f'(x,y) = f_y'(x)$.

We Calderón-Zygmund decompose f' now with respect to α as above, yielding the associated sets $Q'_{x,j,\alpha}$, E'_α. Note that $E_\alpha \supseteq E'_\alpha$. So without loss of generality we may assume $f = f'$. Hence, without loss of generality, $f(x_1,y) \geq f(x_2,y)$ if $0 \leq x_1 < c \leq x_2 \leq 1$, and $\int_0^1 f(x,y)\,dy > \alpha$ if and only if $x < c$.

Let $f_1 = f \cdot \chi_{[0 \leq x < c]}$, $f_2 = f \cdot \chi_{[c \leq x \leq 1]}$. So $f = f_1 + f_2$. Let $\tilde{f}_1$ be a rearrangement of f_1 such that, for each $y \in [0,1]$, $f_1(\cdot,y)$ and $\tilde{f}_1(\cdot,y)$ are equidistributed and $\tilde{f}_1(x,y)$ is nonincreasing in x. Define $\tilde{f}_2$ to be the rearrangement of f_2 within $\{Q \cap \{(x,y) : c \leq x \leq 1\}\}$ such that, for each $y \in [0,1]$, the functions $f_2(\cdot,y)$ and $\tilde{f}_2(\cdot,y)$ are equidistributed and such that $\tilde{f}_2(x,y)$ is nonincreasing in x in $\{x : c \leq x \leq 1\}$. Now $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. Let $E_{1,\alpha} = \cup_{\substack{x \in [0,c) \\ j \in \mathbf{Z}_+}} (x \times Q_{x,j,\alpha})$, $E_{2,\alpha} = \cup_{\substack{x \in [c,1] \\ j \in \mathbf{Z}_+}} (x \times Q_{x,j,\alpha})$. Define $\tilde{E}_{1,\alpha}$ and $\tilde{E}_{2,\alpha}$ similarly. Note that $\left|\tilde{E}_{1,4\alpha}\right| \leq 2\left|E_{1,\alpha}\right|$ trivially (as $Q_{x,1,\alpha} = [0,1]$ if $x < c$) and $\left|\tilde{E}_{2,4\alpha}\right| \leq 2\left|E_{2,\alpha}\right|$ by the special case argument, since $\int_0^1 f(x,y)\,dy \leq \alpha$ for $x \geq c$. As $|E_\alpha| = |E_{1,\alpha}| + |E_{2,\alpha}|$ and $\left|\tilde{E}_\alpha\right| = \left|\tilde{E}_{1,\alpha}\right| + \left|\tilde{E}_{2,\alpha}\right|$, we see that $\left|\tilde{E}_{4\alpha}\right| \leq 2\left|E_\alpha\right|$, as desired. □

Theorem 9. *Suppose f is a measurable function supported on Q. Then*

$$c\int_Q M_{HL} f \leq \int_Q M_x f + \int_Q M_y f \leq C\int_Q M_{HL} f \tag{24}$$

for some universal constants $0 < c, C < \infty$.

Proof. From [4] we realize that $\int_Q M_x f \lesssim \int_Q M_{HL} f$ and $\int_Q M_y f \lesssim \int_Q M_{HL} f$. Hence it suffices to show that $\int_Q M_{HL} f \lesssim \int_Q M_x f + \int_Q M_y f$. We may assume without loss of generality that f is nonnegative. Let $\tilde{f}(x,y)$ be the function

supported on Q which is nonincreasing in x and such that, for each $y \in [0,1]$, $\tilde{f}(\cdot, y)$ and $f(\cdot, y)$ are equidistributed. Let $f^*(x,y)$ be the function supported on Q which is nonincreasing in y and such that, for each $x \in [0,1]$, $f^*(x,\cdot)$ and $\tilde{f}(x,\cdot)$ are equidistributed. Note that $f^*(x_1,y_1) \geq f^*(x_2,y_2)$ whenever $0 \leq x_1 \leq x_2 \leq 1$, $0 \leq y_1 \leq y_2 \leq 1$. As $\int_Q M_{HL} f^* \sim \int_Q M_{HL} f$, $\int_Q M_y f^* \sim \int_Q M_y \tilde{f}$, and $\int_Q M_x \tilde{f} \sim \int_Q M_x f$ by Stein's $L \log L$ result [5], we see that

$$\begin{aligned}
\int_Q M_{HL} f &\sim \int_Q M_{HL} f^* \\
&\lesssim \int_Q M_x f^* + \int_Q M_y f^* \qquad \text{(Lemma 7)} \\
&\lesssim \int_Q M_x \tilde{f} + \int_Q M_y \tilde{f} \qquad \text{(Lemma 8)} \\
&\lesssim \int_Q M_x f + \int_Q M_y f \qquad \text{(Lemma 8),}
\end{aligned}$$

as desired. □

Building upon these ideas, more sophisticated applications of Córdoba-Fefferman collections are used collectively in [2] and [3] to prove that if f is a function supported on Q such that $\int_Q M_y M_x f < \infty$ but $\int_Q M_x M_y f = \infty$, then there exists a set A of finite measure in $\mathbf{R}^2$ such that $\int_A M_S f = \infty$.

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