LIMITATIONS OF ROBUST STABILITY OF A LINEAR DELAYED FEEDBACK CONTROL

D. DMITRISHIN, P. HAGELSTEIN, A. KHAMITOVA, AND A. STOKOLOS

Abstract. In this paper we consider the issue of robust stability of a linear delayed feedback control (DFC) mechanism. In particular we consider a DFC for stabilizing fixed points of a smooth function $f : \mathbb{R}^m \to \mathbb{R}^m$ of the form

$$x(k+1) = f(x(k)) + u(k) \ ,$$

where $u(k) = -\sum_{j=1}^{N-1} \epsilon_j (x(k-j) - x(k-j+1))$. We associate to each fixed point of f an explicit polynomial whose Schur stability corresponds to the local asymptotic stability of the DFC at that fixed point. This polynomial is the characteristic polynomial of the Jacobian matrix of an auxiliary map from $\mathbb{R}^{mN}$ to $\mathbb{R}^{mN}$ and may be given in terms of the eigenvalues of the Jacobian of f at the fixed point. This enables us to evaluate the robustness of the control by considering over what possible sets of eigenvalues of the Jacobian of f the associated characteristic polynomials are Schur stable. We will show that, for a given control of the above form, stability is guaranteed only if the set of eigenvalues lies in a set in $\mathbb{C}$ of diameter less than or equal to 16 and whose connected components all have diameters less than or equal to 4. An explicit example indicating the sharpness of the diameter of connected components is provided.

1. Introduction

The control of chaotic systems is one of considerable interest in the fields of engineering, mathematics, and physics. In the seminal paper [12], Grebogi, Ott, and Yorke observed that chaotic systems may frequently be stabilized by small time-dependent perturbations. Subsequent papers exploring specific control mechanisms for stabilizing chaotic systems included [3, 15]. One method of control of particular interest is the delayed feedback control (DFC) mechanism introduced by Pyragas in [13]. The control in the Pyragas scheme is essentially a multiple of the difference between the current and one period delayed states of the system. Salient advantages of this scheme are that the control term vanishes if the system is already in steady state and that the control term tends to zero as the system approaches a steady state. This DFC control mechanism features many applications, ranging from the stabilization of the modulation index of lasers to the suppression of pathological brain rhythms [1, 14].

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Motivated by the Pyragas scheme and by ideas of De Sousa Vieira and Lichtenberg [17], Dmitrishin and Khamitova considered the system

$$x(k+1) = f(x(k)) + u(k)$$

closed by the *non-linear control*

$$u(k) = (a_1 - 1)f(x(k)) + a_2 f(x(k-1)) + \cdots + a_N f(x(k-(N-1)) \ ,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. The characteristic polynomial associated to this control is given by

$$p(\lambda) = \lambda^N - \mu(a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) \ ,$$

where μ is the derivative of f at the fixed point (the multiplier for the corresponding dynamical system.) In the paper [4], Dmitrishin and Khamatova showed that, given any arbitrarily large integer $-M$, there exists N and coefficients $a_1, \ldots, a_N$ such that the above polynomial is Schur stable provided $\mu \in (-M, 1)$. Modifications of this non-linear control for locally stabilizing T-orbits of a function f were considered by Dmitrishin et al. in the paper [5].

In the present paper we consider the system

$$x(k+1) = f(x(k)) + u(k)$$

closed by the *linear delayed feedback control* given by

$$u(k) = -\sum_{j=1}^{N-1} \epsilon_j (x(k-j) - x(k-j+1)) \ , \tag{1.1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth function. At a given fixed point of f we let $\mu_1, \ldots, \mu_m$ be the eigenvalues of the Jacobian of f at that point. The mN'th degree polynomial whose Schur stability guarantees the local stability of this control at the fixed point is given by

$$\chi(\lambda) = \prod_{j=1}^{m} (\lambda^N + (-\mu_j + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N) \ ,$$

where $a_1 = -\epsilon_1$, $a_j = \epsilon_{j-1} - \epsilon_j$ for $j = 2, \ldots, N-1$, and $a_N = \epsilon_{N-1}$. Note that $a_1 + \cdots + a_N = 0$. The derivation of the explicit form of this polynomial is of very recent vintage, following from the work of the authors in [5, 9]. It is important to observe that a bijective correspondence holds between the set of coefficents $\epsilon_1, \ldots, \epsilon_{N-1}$ and the set of coefficents $a_1, \ldots, a_{N-1}$, and that by finding coefficients $a_1, \ldots, a_{N-1}$ for which $\chi(\lambda)$ is Schur stable we effectively find corresponding coefficents $\epsilon_1, \ldots, \epsilon_{N-1}$ for which $u(k)$ is a suitable control.

We wish to emphasize that the linear delayed feedback control mechanism associated to (1.1) is a very natural one. The reason that a rigorous mathematical treatment of issues surrounding asymptotic stability associated to this control has not appeared until now was largely due to the difficulty arriving at the explicit formulation of the polynomial $\chi(\lambda)$. There are of course other linear control mechanisms; a detailed analysis of these would require finding an explicit form of the corresponding characteristic polynomials. At the present time the construction and analysis of the Schur stability of these polynomials seem to be in general a difficult problem best treated on a case-by case basis.

Now, given the previously mentioned result of Dmitrishin and Khamitova, one might suspect that, provided all the eigenvalues μ_j lie in the interval $(-M, 1)$, there should exist N and $a_1, \ldots, a_N$ such that the above polynomial $\chi(\lambda)$ is Schur stable. It comes as a surprise to us that this is false. We shall see that, given N and $a_1, \ldots, a_N$, the collection of possible points $\mu_1, \ldots, \mu_m$ for which $\chi(\lambda)$ is Schur stable is uniformly limited in size. In particular, we prove the following:

Theorem 1. *Given N and $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, let M_a denote the set of values of μ for which the polynomial*

$$\chi_\mu(\lambda) = \lambda^N + (-\mu + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N$$

is Schur stable. Then the diameter of M_a is less than or equal to 16.

More precise information regarding the size of connected components of M_a is given by the following.

Theorem 2. *Let M_a be defined as above. Let $M_a = \cup M_a^{(j)}$, where $\{M_a^{(j)}\}$ denote the connected components of M_a. Then for every j, the diameter of $M_a^{(j)}$ is less than or equal to 4.*

The sharpness of Theorem 2 is provided by the following.

Theorem 3. *Suppose $-3 < a < b < 1$. Then there exists $\epsilon \in \mathbb{R}$ such that, for* every *$\mu \in (a, b)$, the polynomial*

$$\lambda^2 + (-\mu - \epsilon)\lambda + \epsilon$$

is Schur stable.

An immediate corollary of Theorems 1 and 2 is the following.

Corollary 1. *Given N and $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, let $\tilde{M}_a$ denote the smallest set in $\mathbb{C}$ containing every value $\mu_1 \ldots, \mu_m$ of every collection $\{\mu_j\}_{j=1}^m$ such that*

$$\chi(\lambda) = \prod_{j=1}^{m} (\lambda^N + (-\mu_j + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N)$$

is Schur stable. Then the diameter of $\tilde{M}_a$ is less than or equal to 16. Moreover, if $\tilde{M}_a = \cup \tilde{M}_a^{(j)}$, where every $\tilde{M}_a^{(j)}$ is connected, then the diameter of each $\tilde{M}_a^{(j)}$ is less than or equal to 4.

We remark that, especially due to the ease of computation associated to the control (1.1), the analysis of local stability estimates in scenarios where a range of multipliers is present is highly desirable. For example, we may wish to establish fixed points of a function, say, $\sin(14x^2 - e^x)$ on $(-1, 1)$, with *a priori* unknown multipliers that nonetheless lie in a given bounded range of values. An another example, we may wish to stabilize a system associated to a function $f : \mathbb{R}^m \to \mathbb{R}^m$ whose Jacobian at a given equilibrium point has distinct eigenvalues. The results of this paper indicate that, if the set of possible multipliers associated to f has a diameter exceeding 16 or if an individual connected component of the set of possible multipliers has diameter exceeding 4, the control (1.1) should in general *not* be used to stabilize the associated system. This strongly motivates the use of *non-linear* controls in this type of situation, and the reader is encouraged to consult the related paper [6,7] in this regard.

We recognize that these theorems cast a somewhat depressing aspect on the paper in that they explicitly point out unavoidable limitations on the control (1,1). Note, that this situation is not exceptional rather quite common, c.f. [18]. Nonetheless, the results provided by these theorems are exceedingly practical as we now know not to use this control, natural as it is, to try to provide asymptotic stability for functions with unknown multipliers lying in sufficiently large regions.

In the subsequent section we will provide proofs of these theorems. The proofs will rely on classical but powerful results in complex analysis, in particular taking advantage of Koebe's One-Quarter Theorem. In the third section we will consider examples illustrating the limitations of stability of the linear control theory under consideration. In the closing section we list open problems and suggested avenues of further research.

2. Proofs of limitations of robust stability

We now provide proofs of Theorems 1, 2, and 3 and Corollary 1.

Proof of Theorem 1. Note that if M_a is empty then the result trivially holds. So we may assume without loss of generality that there exists a value μ_0 for which

$$\chi_{\mu_0}(\lambda) = \lambda^N + (-\mu_0 + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N$$

is Schur stable. We now consider the polynomial $\chi_{\mu_0+\Delta\mu}(\lambda)$ defined by

$$\chi_{\mu_0+\Delta\mu}(\lambda) = \lambda^N + (-\Delta\mu - \mu_0 + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N \ .$$

To prove the theorem, it suffices to show that the set of values $\Delta\mu$ for which $\chi_{\mu_0+\Delta\mu}(\lambda)$ is Schur stable has diameter less than or equal to 16. Note that we may reexpress $\chi_{\mu_0+\Delta\mu}(\lambda)$ as

$$\chi_{\mu_0+\Delta\mu}(\lambda) = \lambda^N - \Delta\mu\lambda^{N-1} + p(\lambda) \ ,$$

where $p(z) = (a_1 - \mu_0)z^{N-1} + a_2 z^{N-2} + \cdots + a_N$. We define the auxiliary polynomial $q(z)$ by

$$q(z) = (a_1 - \mu_0)z + \cdots + a_N z^N$$

and the function $\Phi(z)$ by

$$\Phi(z) = \frac{z}{1+q(z)} \ .$$

We now take advantage of a useful observation of Solyanik [16].

Lemma 1 (Solyanik)**.** *The polynomial* $\chi_{\mu_0+\Delta\mu}(\lambda) = \lambda^N - \Delta\mu\lambda^{N-1} + p(\lambda)$ *is Schur stable if and only if*

$$\frac{1}{\Delta\mu} \in \bar{\mathbb{C}} \backslash \Phi(\bar{\mathbb{D}}) \ ,$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$*,* $\{\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$*, and* $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$*.*

Proof. The polynomial $\chi_{\mu_0+\Delta\mu}(\lambda)$ is Schur stable if and only if all of its roots lie in the unit disk $\mathbb{D}$, i.e. $\chi_{\mu_0+\Delta\mu}(\lambda) \neq 0$ for all $\lambda \in \bar{\mathbb{C}} \backslash \mathbb{D}$. This is equivalent to

$$\frac{1}{\Delta\mu} \neq \frac{\frac{1}{\lambda}}{1 + \frac{1}{\lambda^N} p(\lambda)} \ , \ \lambda \in \bar{\mathbb{C}} \backslash \mathbb{D}$$

or equivalently

$$\frac{1}{\Delta\mu} \neq \frac{z}{1+q(z)} \ , \ z \in \bar{D} \ .$$

Hence a polynomial $\chi_{\mu_0+\Delta\mu}(\lambda)$ is Schur stable if and only if $\frac{1}{\Delta\mu} \notin \Phi(\bar{\mathbb{D}})$, or equivalently $\frac{1}{\Delta\mu} \in \bar{\mathbb{C}} \backslash \Phi(\bar{\mathbb{D}})$, as desired.

□

We now state a result in classical complex analysis due to Carathéodory [2].

Lemma 2 (Carathéodory)**.** *Let*

$$\phi(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

be an analytic function in the unit disc $\mathbb{D}$ *that does not vanish anywhere in* $\mathbb{D}$ *except at the origin. Then* $\phi(\mathbb{D})$ *contains the disc in* $\mathbb{C}$ *about the origin of radius* $|c_1|/16$*.*

Now, observe that $\Phi(z)$ is of the form

$$\Phi(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

and hence by Lemma 2 we have that $\Phi(\mathbb{D})$ contains the open disc in $\mathbb{C}$ of radius $1/16$. By Lemma 1 we then realize that $\chi_{\mu_0+\Delta\mu}$ is Schur stable only if $|\Delta\mu| \leq 16$. Note that as μ_0 is an *arbitrary* point in M_a, we then have that that the diameter of M_a is less than or equal to 16, concluding the proof of Theorem 1.

□

Proof of Theorem 2. Let μ_0 be an arbitrary point in $M_a^{(j)}$ for some fixed j, and define $\chi_{\mu_0+\Delta\mu}$ and Φ as in the proof of Theorem 1. Since $\chi_{\mu_0}(\lambda) = \lambda^N + p(\lambda)$ is Schur stable, all the poles of the function $\Phi(z)$ lie outside $\bar{\mathbb{D}}$. Hence $\Phi(z)$ is analytic on $\mathbb{D}$. Hence $\Phi(\mathbb{D})$ is an open set. Now, $\Phi(\mathbb{D})$ is not necessarily simply connected, but there does exist a minimial open simply connected set containing $\Phi(\mathbb{D})$ that we designate as $\Phi^s(\mathbb{D})$. Note $\Phi^s(\mathbb{D})$ is not all of $\mathbb{C}$ as

$\Phi(z)$ is bounded on $\bar{\mathbb{D}}$). Moreover we have $0 \in \Phi^s(\mathbb{D})$.

Lemma 3. *The set $\Phi^s(\mathbb{D})$ contains a disc of radius $1/4$ about the origin.*

Proof. By the Riemann Mapping Theorem there exists bijective holomorphic map ϕ from $\mathbb{D}$ onto $\Phi^s(\mathbb{D})$, where $\phi(0) = 0$, $\phi'(0) > 0$. Hence $\phi^{-1}(\Phi(\mathbb{D})) \subset \mathbb{D}$. So the function $F : \mathbb{D} \to \mathbb{D}$ defined by $F(z) = \phi^{-1}(\Phi(z))$ satisfies the inequality $|F(z)| < 1$ for every $z \in \mathbb{D}$. By the Schwarz lemma we then have $|F'(0)| \leq 1$. We compute that

$$F'(0) = (\phi^{-1})'(0) \cdot \Phi'(0) = (\phi^{-1})'(0) \ .$$

This implies that $(\phi^{-1})'(0) \leq 1$, $\phi'(0) \geq 1$. By the Koebe One-Quarter Theorem we have that $\phi(\mathbb{D})$ contains an open disk of radius $\frac{\phi'(0)}{4}$ about the origin and hence one of radius $1/4$ about the origin. $\square$

Now, since $a_1, \ldots, a_N$ are real, the set $\Phi(\mathbb{D})$ is symmetric about the real axis in the complex plane, and hence its complement contains only one unbounded component, which happens to be the complement of $\Phi^s(\mathbb{D})$. Note then that by Lemma 1 we have that, if $\mu_0 + \Delta\mu$ lies in the same component of M_a as μ_0, we must have that $1/\Delta\mu$ lies in the complement of $\Phi^s(\mathbb{D})$. As $\Phi^s(\mathbb{D})$ contains an open disc of radius $1/4$ about the origin, we see then that $|\Delta\mu| \leq 4$. As μ_0 is an arbitrary point in $M_a^{(j)}$, we have that the diameter of $M_a^{(j)}$ is less than or equal to 4, as desired. $\square$

Proof of Theorem 3. Let $\epsilon \in (-\frac{1+a}{2}, 1)$. Using the quadratic formula and considering multiple cases, one may show that for every $\mu \in (a, b)$ the roots of $\lambda^2 + (\mu - \epsilon)\lambda + \epsilon$ lie in the unit disc $\mathbb{D}$. We omit the somewhat lengthy but straightforward details. $\square$

3. Remarks and Examples

In this section we will provide some remarks and examples that will help the reader, especially a non-expert in control theory, to better appreciate the above theorems and to place them into context. Now, in this paper we are interested in, given a positive integer N and real numbers $a_1, \ldots, a_N$ such that $\sum_{j=1}^N a_j = 0$, for which μ we have that

$$\chi_\mu(\lambda) = \lambda^N + (-\mu + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N$$

is Schur stable, i.e. for which values of λ do all the roots of the above polynomial lie in the unit disc of $\mathbb{C}$? Some simple but important observations are in order here. Note that χ_μ has the factorization

$$\chi_\mu(\lambda) = (\lambda - r_1)(\lambda - r_2) \cdots (\lambda - r_N) \ .$$

If χ_μ is Schur stable, then none of its roots are going to lie in $[1, \infty)$. If μ were real, since $\chi_\mu(1) = 1 - \mu$ and $\chi_\mu(\lambda)$ tends to infinity and λ tends to infinity, by the mean value theorem we must have that $1 - \mu > 0$, i.e. $\mu < 1$. So for real μ we have that χ_μ is not going to be Schur stable unless $\mu < 1$. We have a lower bound for μ as well. To see this, note that if χ_μ

is Schur stable, then by the above factorization we have that $\chi_\mu(1) < 2^N$, seen by plugging in -1 for all of the r_j. As $\chi_\mu(1) = 1 - \mu$, this yields $1 - \mu < 2^N$.

On the other hand, suppose we had $0 < 1 - \mu < 2^N$. Note that we could certainly find r_j within $(-1, 1)$ such that

$$(1 - r_1)(1 - r_2) \cdots (1 - r_N) = 1 - \mu \ ;$$

in fact we could set all of them equal to $1 - (1 - \mu)^{1/N}$. Noting that

$$\chi_\mu(1) = 1 - \mu + \sum_{j=1}^{N} a_j = \prod_{j=1}^{N} (1 - r_j) \ ,$$

we then see there would exist $a_1, \ldots, a_N$ with $\sum_{j=1}^{N} a_j = 0$ such that χ_μ is Schur stable.

We see then, provided that $0 < 1 - \mu < 2^N$, we may find $a_1, \ldots, a_N$ with $\sum_{j=1}^{N} a_j = 0$ such that χ_μ is Schur stable. This is quite nice in that we may then use a linear control of the form (1.1) to stabilize a particular equilibrium point of a function, regardless of how negative the multiplier of the function is at that point. (Of course, N must grow with the size of the multiplier.) Unfortunately, given N and $a_1, \ldots, a_N$, the above considerations provide little information regarding what equilibrium points with *other* multipliers are stabilized by the control.

This is the context of the results of our paper. In the above theorems we have shown that, given N and $a_1, \ldots, a_N$, the linear control (1.1) in general will provide stability only to equilibrium points whose multipliers lie in a bounded region; in particular whose multipliers lie in the complex plane of set diameter less than 16 and whose connected components all have a diameter less than 4.

We now illustrate the above comments with three instructive examples.

Example 1: Consider the function $f(x) = h \sin(\pi x)$. All derivatives of f lie in the interval $[-\pi h, \pi h]$. If $h \in (-h_0, \frac{1}{\pi})$ for $\frac{1}{\pi} < h_0 < 1$ then the associated set M of multipliers of f would be a subset of $(-\pi h_0, 1)$. If $h_0 \leq \frac{3}{\pi}$, then by Theorem 3 all of the equilibrium points of the system

$$x_{n+1} = h \sin(\pi x_n)$$

can be locally stabilized by a control of the form

$$u = -\epsilon_1 (x_{n-1} - x_n) \ .$$

Alternatively, corresponding to a situation where $\frac{3}{\pi} < h_0 \leq 1$, any fixed point in the above system with multiplier $\mu \in (-\pi, -3)$ can be locally stabilized by a control of the form

$$u = -\epsilon_1(\mu)(x_{n-1} - x_n) - \epsilon_2(\mu)(x_{n-2} - x_{n-1}) \ .$$

Note that in this case we have $1-\mu < 1+\pi < 2^3$, so by the previous remarks there exist a_1, a_2, a_3 with $a_1 + a_2 + a_3 = 0$ and such that

$$\chi_\mu(\lambda) = \lambda^3 + (-\mu + a_1)\lambda^2 + a_2\lambda + a_3$$

is Schur stable. $\epsilon_1, \epsilon_2, \epsilon_3$ are given by the formulas $a_1 = \epsilon_1$, $a_2 = \epsilon_1 - \epsilon_2$, $a_3 = \epsilon_2$. However, there does not exist N such that all fixed points of the system may be locally stabilized by a control of the form

$$u = -\sum_{j=1}^{N-1} \epsilon_j (x_{k-j} - x_{k-j+1}) \ .$$

Example 2: In their paper on Kharitonov's Stability Criterion, Hollot and Bartlett considered the polynomial $F(z) = z^4 + a_1 z^3 + 1.35z^2 + 0.243z - 0.2916$ and noted that $F(z)$ is Schur stable for $a_1 = -2.3$ or $a_1 = 1.7$, but not Schur stable at $a_1 = -1.3$ ([8]; see also [10]). The observations of Solyanik in Lemma 1 permit us to relatively easily establish *exactly* for which a_1 the above polynomial is Schur stable. Define the auxiliary function $\tilde{\Phi}$ by

$$\tilde{\Phi}(z) = \frac{z}{1 + 1.35z^2 + 0.243z^3 - 0.2916z^4} \ .$$

Using direct algebraic substitution, Lemma 1 implies that $F(z)$ is Schur stable if and only if

$$\frac{-1}{a_1} \in \mathbb{C} \backslash \tilde{\Phi}(\bar{\mathbb{D}}) \ .$$

The values of a_1 for which $F(z)$ is Schur stable are the shaded regions in the following figures. Note that the diameter of the values of a_1 for which $F(z)$ exceeds 4 (but is less than 16), although the connected components of the region of stability all have diameter less than 4. The contours plotted in Figures 1 - 3 correspond to the image of $-\frac{1}{\tilde{\Phi}(z)}$ when acting on the boundary of the unit disc in $\mathbb{C}$. Finally, note that the value -2.3 observed by Hollot and Bartlett lies in the left component illustrated by Figure 2; the value 1.7 lying (just barely!) in the right component illustrated by Figure 3.

It is informative to consider how this example relates to the construction of a particular control. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (\mu_1 x, \mu_2 y)$. Set $a_1 = -1.3014, a_2 = 1.35, a_3 = 0.243$, and $a_4 = -0.2916$. Note $a_1 + \cdots + a_4 = 0$. Setting $\epsilon_1 = -a_1$, $a_j = \epsilon_{j-1} - \epsilon_j$ for $j = 2, 3$, we yield $\epsilon_1 = 1.3014$, $\epsilon_2 = -0.0486$, and $\epsilon_3 = -0.2916$. The system associated to f given by

$$x_{n+1} = \mu_1 x_n$$
$$y_{n+1} = \mu_2 y_n$$

and its equilibrium may be locally stabilized by the control

$$u_n = \begin{pmatrix} u_n^{(1)} \\ u_n^{(2)} \end{pmatrix} = -\sum_{j=1}^{3} \epsilon_j \begin{pmatrix} x_{n-j} - x_{n-j+1} \\ y_{n-j} - y_{n-j+1} \end{pmatrix}$$

provided μ_1 and μ_2 lie in the set of values of μ for which

$$\lambda^4 + (-\mu + a_1)\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

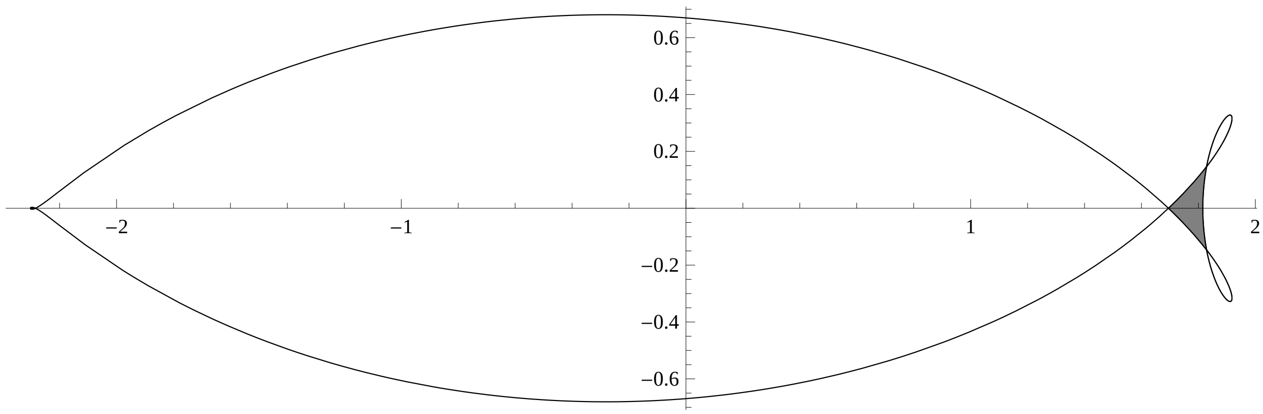

FIGURE 1. Region of Stability

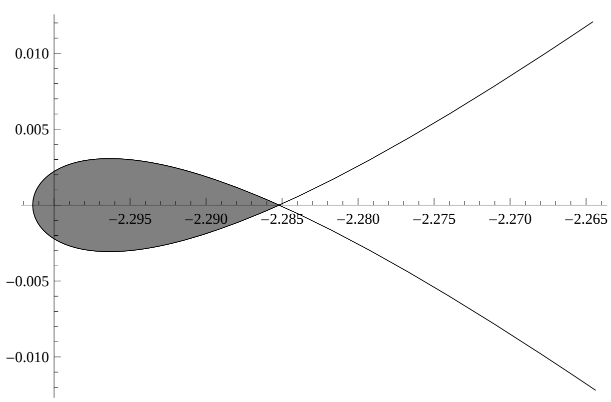

FIGURE 2. Left Component of the Region

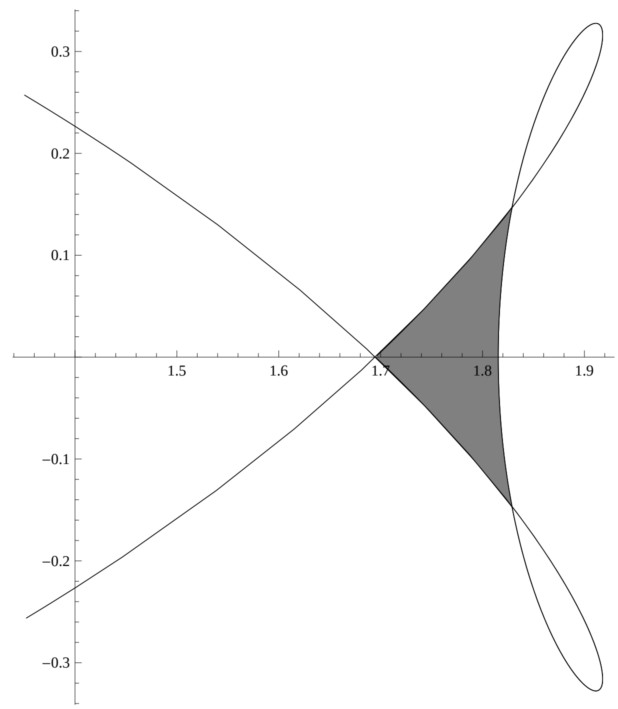

FIGURE 3. Right Component of the Region

is Schur stable. Now, by the previous observations (translating a_1 to $a_1 - \mu$), we have that $\lambda^4 + (-\mu + a_1)\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$ is Schur stable provided $-\mu + a_1$, in particular $-\mu - 1.3014$, lies in the shaded region indicated in Figures 1-3. So the above system could be stabilized for, say, $\mu_1 = -3.1014$, $\mu_2 = 0.9987$. It is worthwhile to note that the control in this case is able to stabilize the trivial equilibrium point of the above system in some scenarios where the associated multipliers differ by more than 4. However, there is no linear control that locally stabilizes the equilibrium of this system where the multipliers are able to freely range over an entire interval in $\mathbb{R}$ of length exceeding 4.

Example 3: We consider now the system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = h \begin{pmatrix} \sin(x_n + y_n) \\ \sin(y_n + z_n) \\ \sin(z_n + x_n) \end{pmatrix}$$

where h lies in a set $H \subset \mathbb{R}$. This system has a trivial equilibrium that corresponds to the set of multipliers

$$M = \{he^{i\frac{\pi}{3}}, he^{-i\frac{\pi}{3}}, 2h\} .$$

If $H = \{-h_0\}$ then for $|h_0| > \frac{16}{\sqrt{3}} \approx 9.24$ the diameter of the set M exceeds 16 and hence by Theorem 1 there is no control of the form (1.1) that locally stabilizes the trivial equilibrium of the system. Moreover, by Theorem 2 we realize that, if $H = (-h_0, 0)$, $h_0 > 2$, there is no a control of the form (1.1) that locally stabilizes the above system for all $h \in (-h_0, 0)$.

4. Open Problems

We conclude by indicating the following problems that we believe to be suitable directions of further research.

Problem 1: It is interesting that the constant 4 in Theorem 2 is known to be sharp, as is shown by Theorem 3, although we do not know if the constant 16 in Theorem 1 is sharp. In particular, we ask:

Given N and $a = (a_1, \dots, a_N)$ such that $a_1 + \dots + a_N = 0$, let M_a denote the set of values of μ for which the polynomial

$$\chi_\mu(\lambda) = \lambda^N + (-\mu + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \dots + a_N$$

is Schur stable. What is the smallest value of ρ such that we are guaranteed that the diameter of M_a is less than or equal to ρ?

Problem 2: This paper has considered the issue of robust stability of a linear control designed to stabilize *fixed points* of a function $f : \mathbb{R}^n \to \mathbb{R}^n$. It is natural to consider the linear control counterpart of the nonlinear control considered by Dmitrishin et al. in [5] of the form

$$u(k) = (a_1 - 1)f(x(k)) + a_2 f(x(k-T)) + \dots + a_N f(x(k-(N-1)T)) ,$$

with $a_1 + \dots + a_N = 1$, designed to stabilize *T-cycles of f*. This linear control is of the form

$$u(k) = -\sum_{j=1}^{N-1} \epsilon_j (x((k-j)T) - x((k-j+1)T)) .$$

However, at the present time we have not been able to find a transparent form of the characteristic polynomial associated to this control, much less analyze issues of stability for this control. We ask:

What is the characteristic polynomial associated to the linear control above, expressed in a transparent way in terms of $a_1, \ldots, a_N$? Given N and $a_1, \ldots, a_N$, for what values of μ_j is the corresponding control polynomial Schur stable?

We wish to emphasize that the results of this paper highlight that, in terms of applications, caution should be used in implementing a linear control of the form 1.1, given the now-proven limitations on multipliers of equilibrium points that may be stabilized by this control. In that regard the problems indicated above are admittedly of theoretical interest. We also remark that recent research of some of the authors indicate that the limitations of stability associated to the linear controls considered here may be successfully bypassed by use of *nonlinear controls*, and we strongly encourage the interested reader to consult [6,7] in this regard.

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Odessa National Polytechnic University, 1 Shevchenko Avenue, Odessa 65044, Ukraine
E-mail address: `dmitrishin@opu.ua`

Department of Mathematics, Baylor University, Waco, Texas 76798
E-mail address: `paul_hagelstein@baylor.edu`

Georgia Southern University, Department of Mathematical Sciences, 65 Georgia Avenue, Statesboro, Georgia 30460-8093
E-mail address: `anna_khamitova@georgiasouthern.edu`

Georgia Southern University, Department of Mathematical Sciences, 65 Georgia Avenue, Statesboro, Georgia 30460-8093
E-mail address: `astokolos@georgiasouthern.edu`