PROBLEMS IN INTERPOLATION THEORY RELATED TO THE ALMOST EVERYWHERE CONVERGENCE OF FOURIER SERIES

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ABSTRACT. A well-known conjecture in harmonic analysis is that the sequence of partial Fourier sums of a function in $L \log L(\mathbb{T})$ converges almost everywhere. The purpose of this expository paper is to discuss connections between this conjecture and recent developments in interpolation theory regarding sublinear translation invariant restricted weak type operators. Open problems in interpolation theory motivated by these results will also be presented.

Let f be a measurable function supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The *n'th Fourier coefficient of* f is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta \ .$$

The *N'th Dirichlet sum of* f (frequently called the *N'th partial Fourier sum of* f) is given by

$$S_N f(e^{i\theta}) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}.$$

One of the fundamental questions in harmonic analysis is: In what sense does $S_N f$ converge to f as N tends to infinity? In terms of convergence in *norm* we have the following classical results due to Marcel Riesz and Antoni Zygmund ([12], [22]):

Theorem 1. *If $f \in L^p(\mathbb{T})$, $1 < p < \infty$, then*

$$\lim_{N\to\infty} \|S_N f - f\|_p = 0.$$

Moreover, if $\int_0^{2\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})| d\theta < \infty$, then

$$\lim_{N\to\infty} \|S_N f - f\|_1 = 0.$$

In terms of *almost everywhere* convergence, the best known result is the following one due to Carleson:

2000 *Mathematics Subject Classification.* Primary 42B35.

P. A. Hagelstein's research was partially supported by the Baylor University Summer Sabbatical Program.

Theorem 2 ([2]). *If $f \in L^2(\mathbb{T})$, then the sequence of functions $\{S_N f\}$ converges to f almost everywhere.*

Hunt provided the following improvement upon Carleson's result:

Theorem 3 ([6]). *If $f \in L^p(\mathbb{T})$, $1 < p < \infty$, then $\{S_N f\}$ converges to f almost everywhere. Moreover, if $\int_0^{2\pi} |f(e^{i\theta})|(\log^+ |f(e^{i\theta})|)^2 d\theta < \infty$, then $\{S_N f\}$ converges to f almost everywhere.*

Sjölin improved upon Hunt's result by proving the following:

Theorem 4 ([14]). *If $\int_0^{2\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})| \log^+(\log^+(|f(e^{i\theta})|))d\theta < \infty$, then $\{S_N f\}$ converges to f almost everywhere.*

This result by Sjölin was the a.e. convergence world record holder for nearly thirty years, finally improved upon by Antonov (see also the subsequent paper [15] by Sjölin and Soria) with the following:

Theorem 5 ([1]). *If $\int_0^{2\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})| \log^+(\log^+(\log^+(|f(e^{i\theta})|)))d\theta < \infty$, then $\{S_N f\}$ converges to f almost everywhere.*

Many mathematicians believe the optimal a.e. convergence result is given by the following:

Conjecture 1. *If $\int_0^{2\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})| d\theta < \infty$, then $\{S_N f\}$ converges to f almost everywhere.*

Now, when Hunt obtained his L^p and $L(\log L)^2$ a.e. convergence results, he showed that the Carleson maximal operator, defined by

$$Mf(e^{i\theta}) = \sup_N |S_N f(e^{i\theta})|,$$

satisfies the estimate

$$(*) \qquad \left|\{e^{i\theta} : M\chi_E(e^{i\theta}) > \alpha\}\right| \leq \frac{c}{p-1}\left(\frac{\|\chi_E\|_p}{\alpha}\right)^p$$

for any $1 < p < 2$, measurable set E in $\mathbb{T}$, and $\alpha > 0$.

From $(*)$ one can show that, for any $\alpha > 0$ and measurable set E in $\mathbb{T}$,

$$(1) \qquad \left|\{e^{i\theta} : M\chi_E(e^{i\theta}) > \alpha\}\right| \leq \frac{c}{\alpha}\|\chi_E\|_{L\log L(\mathbb{T})}.$$

Recall here that the Orlicz norm $L \log L(\mathbb{T})$ is defined by

$$\|f\|_{L\log L(\mathbb{T})} = \inf\left\{c > 0 : \int_0^{2\pi} \frac{|f(e^{i\theta})|}{c} \log^+\left(\frac{|f(e^{i\theta})|}{c}\right) \leq 1\right\}.$$

One may obtain (1) from $(*)$ by, given a measurable set E in $\mathbb{T}$, taking for p the value $1 + \frac{\|\chi_E\|_{L^1(\mathbb{T})}}{\|\chi_E\|_{L\log L(\mathbb{T})}}$.

Now by standard arguments, to prove Conjecture 1 it is enough to prove the following:

Conjecture 2. *There exists a finite constant C such that*

$$|\{e^{i\theta} : Mf(e^{i\theta}) > \alpha\}| \leq \frac{C}{\alpha}\|f\|_{L\log L(\mathbb{T})}$$

holds for any $\alpha > 0$ and simple function f supported on $\mathbb{T}$.

Note that by (1) we have that the Carleson maximal operator M is of restricted weak type $(L\log L, L^1)$ and to prove Conjectures 1 and 2 we need to prove that, at least when acting on simple functions, M is of weak type $(L\log L, L^1)$. This begs the question: if a sublinear operator is of restricted weak type $(L\log L, L^1)$, must it be of weak type $(L\log L, L^1)$? The answer is no, as indicated by the following example due to Konyagin [8].

Example 1 (Konyagin)**.**

Let $\{E_k\}$ be a collection of disjoint sets in $\mathbb{T}$ such that $|E_k| = 2^{-2^k}$, $k = 1, 2, 3, \ldots$. For each positive integer n, define the sequence of functions $\{g_{n,k}\}_{k=1}^n$ by

$$g_{n,k}(e^{i(\theta+\frac{2\pi k}{n})}) = \frac{1}{|\theta|} \qquad (-\pi \leq \theta < \pi).$$

Let the operator T_n be defined on $L^1(\mathbb{T})$ by

$$T_nf(e^{i\theta}) \;=\; \sum_{k=1}^{n}\left(2^k\int_{E_k}|f(e^{i\theta})|d\theta\right) g_{n,k}(e^{i\theta})\,.$$

One can check that the operators T_n uniformly satisfy the restricted weak type estimate

$$|\{e^{i\theta} : T_n\chi_E(e^{i\theta}) > \alpha\}| \;\leq\; C\frac{\|\chi_E\|_{L\log L(\mathbb{T})}}{\alpha}$$

although there exist $\{f_n\}$, $\{\alpha_n\}$ such that

$$|\{e^{i\theta} : T_nf_n(e^{i\theta}) > \alpha_n\}| \;\geq\; C\log n\,\frac{\|f_n\|_{L\log L(\mathbb{T})}}{\alpha_n}\,.$$

Defining the operator T on $L^1(\mathbb{T})$ by, say,

$$Tf(e^{i\theta}) = \sum_{k=1}^{\infty}\frac{1}{2^k}T_{2^{2^k}}f(e^{i\theta}),$$

we have T is of restricted weak type $(L\log L, L^1)$ but not of weak type $(L\log L, L^1)$.

Note that the operators T_n above are not *translation invariant*. This naturally raises the issue of whether or not we can rescue the situation by imposing on the operators considered the condition of translation invariance. In that regard we have the following conjecture.

Conjecture 3. *Suppose T is a sublinear translation invariant operator acting on $L^1(\mathbb{T})$ such that*

$$|\{e^{i\theta} : |T\chi_E(e^{i\theta})| > \alpha\}| \leq \frac{1}{\alpha}\|\chi_E\|_{L\log L(\mathbb{T})}$$

holds for any measurable set E in $\mathbb{T}$ and $\alpha > 0$. Then there exists a finite constant C such that

$$|\{e^{i\theta} : |Tf(e^{i\theta})| > \alpha\}| \leq \frac{C}{\alpha}\|f\|_{L\log L(\mathbb{T})}$$

holds for any simple function f on $\mathbb{T}$ and $\alpha > 0$.

Note that Conjecture 3 implies Conjecture 1.

Three fundamental issues make the resolution of Conjecture 3 difficult. The first is that it is a statement about the space Weak $L^1(\mathbb{T})$, which is not a Banach space since the weak L^1 "norm" does not satisfy the triangle inequality. (For example, note that $\|\frac{1}{x}\|_{WL^1(0,1)} = \|\frac{1}{1-x}\|_{WL^1(0,1)} = 1$, although $\|\frac{1}{x} + \frac{1}{1-x}\|_{WL^1(0,1)} = 4$.) The second is that the Orlicz class $L\log L(\mathbb{T})$ is difficult to work with computationally, and the third is that it is unclear how the translation invariance condition should come into play.

Now, we all realize that in facing a difficult problem it is frequently a good idea to consider analogous problems which are conceptually and computationally easier to deal with. Regarding Conjecture 3 we might then consider the somewhat simpler problem: if a sublinear operator (translation invariant or otherwise) is of restricted weak type (1,1), must it be of weak type (1,1)? Or we might even ask a more general question: what of interest can be said about sublinear operators acting on $L^1(\mathbb{T})$ that are of restricted weak type (1,1)?

Well, suppose T is an operator acting on $L^1(\mathbb{T})$ that is of restricted weak type (1,1) and we wish to know for which normed spaces L_Φ we have $\|Tf\|_{WL^1} \leq c\|f\|_{L_\Phi}$ for any simple function $f \in L_\Phi$. Now, if $f = \sum a_j \chi_{E_j}$, we do have by the sublinearity of T that $|Tf| \leq \sum |a_j T(\chi_{E_j})|$. Again, though, as Weak $L^1(\mathbb{T})$ is not a Banach space we can't simply assert that $\|\sum |a_j T(\chi_{E_j})|\,\|_{WL^1} \leq \sum \|a_j T(\chi_{E_j})\|_{WL^1} \leq c \sum \|a_j \chi_{E_j}\|_1 = c\|f\|_1$. We do have the following result due to E. M. Stein and N. Weiss (see also [7] of Kalton), however.

Theorem 6 ([19])**.** *Suppose that for $j = 1, 2, \ldots,$ $g_j(x)$ is a nonnegative function on a measure space for which $|\{x : g_j(x) > \alpha\}| < \frac{1}{\alpha}$ when $\alpha > 0$. Let $\{c_j\}$ be a sequence of positive numbers with $\sum c_j = 1$, and set*

$K = \sum c_j \log \frac{1}{c_j}$. *Then if* $\alpha > 0$,

$$\left| \left\{ x : \sum_j c_j g_j(x) > \alpha \right\} \right| < \frac{2(K+2)}{\alpha}.$$

With a little work one can use the weak L^1 bound given above for a sum of functions in Weak L^1 to obtain the following (a generalization of this result due to Soria may be found in [16]) :

Theorem 7. *Let* T *be a sublinear operator acting on* $L^1(\mathbb{T})$. *Suppose that, for any measurable subset* E *of* $\mathbb{T}$ *and* $\alpha > 0$,

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha}.$$

Then there exists a finite constant C *such that the inequality*

$$|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \leq C \frac{\|f\|_{L \log L(\mathbb{T})}}{\alpha}$$

holds for any simple function f *supported on* $\mathbb{T}$ *and* $\alpha > 0$.

Now, when we think of the operators that we typically deal with in harmonic analysis that are of restricted weak type (1,1), such as the Hardy-Littlewood maximal operator or the Hilbert transform, they not only map $L \log L(\mathbb{T})$ into Weak $L^1(\mathbb{T})$, they map $L \log L(\mathbb{T})$ into $L^1(\mathbb{T})$ itself (see, e.g., [17]). Of course, the Hardy-Littlewood maximal operator and Hilbert transform are translation invariant, which readily leads us to the statement of the following theorem:

Theorem 8. *If* T *is a translation invariant sublinear operator acting on* $L^1(\mathbb{T})$ *such that*

$$|\{e^{i\theta} : |T\chi_E(e^{i\theta})| > \alpha\}| \leq \frac{|E|}{\alpha}$$

holds for any measurable subset E *of* $\mathbb{T}$ *and* $\alpha > 0$, *then there exists a finite constant* C *such that*

$$\|Tf\|_{L^1(\mathbb{T})} \leq C\|f\|_{L \log L(\mathbb{T})}$$

holds for any simple function f *supported on* $\mathbb{T}$.

I will be sketching a proof of this result, as the proof suggests some open problems in interpolation theory that I find to be of considerable interest. The basic ideas of this proof are extensions of those found in the paper [18] by Stein on limits of sequences of operators. The key lemma for this proof, given below, is that a sublinear translation invariant operator acting on $L^1(\mathbb{T})$ of restricted weak type (1,1) must be of weak type (2,2) when acting on simple functions. For full disclosure, I mention that I published a proof of this lemma in [4] but have subsequently discovered that it also follows from the Nikishin theory as developed by Maurey and Wojtaszczyk in [9], [11], and [20]; see also the related paper [13] by Shteinberg. In particular, this theory yields the result that any sublinear translation invariant operator acting on $L^1(\mathbb{T})$ that maps $L^2(\mathbb{T})$ continuously into $L^0(\mathbb{T})$ (as our operator T does since it maps $L\log L(\mathbb{T})$ into Weak $L^1(\mathbb{T})$) must be of weak type (2,2).

Lemma 1. *Let T be a sublinear translation invariant operator acting on $L^1(\mathbb{T})$. Suppose also for any measurable set E in $\mathbb{T}$ and $\alpha > 0$ that*

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha}\ .$$

Then there exists a finite constant C such the inequality

$$|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \leq C\left(\frac{\|f\|_{L^2(\mathbb{T})}}{\alpha}\right)^2 \tag{2}$$

holds for any $\alpha > 0$ and simple function f supported on $\mathbb{T}$.

Proof. By contradiction. Suppose (2) were false. Then there would exist a sequence of simple functions $\{f_n\}$ and a sequence of sets $\{E_n\}$ such that $|Tf_n(x)| > 1$ if $x \in E_n$, $\sum |E_n| = \infty$, and $\sum \|f_n\|_2^2 < \infty$.

As $\sum \|f_n\|_2^2$ converges, we may find a sequence $\{R_n\}$ of positive numbers such that $R_n \to \infty$ but such that $\sum \|R_n f_n\|_2^2 = D < \infty$.

Now, for each $g \in \mathbb{T}$, we let τ_g denote the translation operator defined by

$$\tau_g f(x) = f(-g + x)\,.$$

As $\sum |E_n| = \infty$, by Stein's modified Borel-Cantelli Lemma in [18] we see that there exists a sequence $\{F_n\}$ of sets in $\mathbb{T}$ such that each F_j is a translate of E_j in $\mathbb{T}$ and such that almost every point of $\mathbb{T}$ belongs to an infinite number of the sets F_j. We associate to each F_j an element $g_j \in \mathbb{T}$ such that

$$\chi_{F_j} = \tau_{g_j}\chi_{E_j}\,.$$

Let M be a positive integer. There exists a positive integer N and a subset $S \subset \mathbb{T}$ of measure greater than 1/2 such that for all x in S there exists an integer j_x such that $1 \leq j_x \leq N$ and

$$M < \left|R_{j_x} T\left(\tau_{g_{j_x}} f_{j_x}\right)(x)\right|.$$

Now, we define the function $h(x,t)$ on $\mathbb{T} \times [0,1]$ by

$$h(x,t) = \sum_{j=1}^{N} R_j \tau_{g_j} f_j(x)\, r_j(t)\,,$$

where $\{r_j(t)\}$ denote the standard Rademacher functions.

If $g(x,t)$ is a measurable function on $\mathbb{T}\times[0,1]$, we define $Tg(x,t)$ by

$$Tg(x,t) = Tg_t(x),$$

where $g_t(x) = g(x,t)$.

Let $x_0 \in S$. For some j such that $1 \le j \le N$ we have that $\left|R_j T\left(\tau_{g_j} f_j\right)(x_0)\right| > M$. We assume without loss of generality that $j = 1$.

Now, if $0 < t < 1$ and t is not of the form $k \cdot 2^j$ for some integers j, k, the sublinearity of T implies that

$$\begin{aligned} M < \left|T\left(R_1\tau_{g_1} f_1\right)(x_0)\right| \le \frac{1}{2}\Bigg[&\left|T\left(R_1\tau_{g_1} f_1(x) + \sum_{j=2}^{N} R_j\tau_{g_j} f_j(x)\, r_j(t)\right)(x_0)\right| \\ &+ \left|T\left(R_1\tau_{g_1} f_1(x) + \sum_{j=2}^{N} R_j\tau_{g_j} f_j(x)\, r_j(1-t)\right)(x_0)\right|\Bigg]. \end{aligned} \tag{3}$$

So $\left|\{t\in[0,1] : |Th(x_0,t)| > M\}\right| \ge 1/2$. As $|S| > \frac{1}{2}$, we then have that

$$\left|\{(x,t)\in\mathbb{T}\times[0,1] : |Th(x,t)| > M\}\right| \ge \frac{1}{4}. \tag{4}$$

Note by the standard orthonormal properties of the Rademacher functions and the Fubini theorem we have

$$\begin{aligned} \|h\|^2_{L^2(\mathbb{T}\times[0,1])} &= \int_{x\in\mathbb{T}}\int_{t=0}^{1}\left(\sum_{j=1}^{N} R_j\tau_{g_j} f_j(x)\, r_j(t)\right)^2 dt\; dx \\ &= \int_{x\in\mathbb{T}}\sum_{j=1}^{N}\left|R_j\tau_{g_j} f_j(x)\right|^2\; dx \\ &= \sum_{j=1}^{N}\left\|R_j\tau_{g_j} f_j(x)\right\|^2_{L^2(\mathbb{T})} \\ &= \sum_{j=1}^{N}\left\|R_j f_j(x)\right\|^2_{L^2(\mathbb{T})} \\ &\le D < \infty. \end{aligned}$$

For our notational convenience, if L^Φ is a normed space on $[0,1]$ and L^Ψ is a normed space on $\mathbb{T}$, we define the mixed norm $\|\cdot\|_{L^\Phi_t(L^\Psi)_x}$ on functions on $[0,1]\times\mathbb{T}$ by

$$\|f(x,t)\|_{L^\Phi_t(L^\Psi)_x} = \left\|\|f(\cdot,t)\|_{L^\Psi(\mathbb{T})}\right\|_{L^\Phi([0,1])}.$$

Note that

$$\begin{aligned} (5) \qquad \|h(x,t)\|_{L^1_t(L\log L)_x} &\leq 10\,\|h(x,t)\|_{L^1_t(L^2)_x} \\ &\leq 10\,\|h(x,t)\|_{L^2_t(L^2)_x} \\ &= 10\,\|h(x,t)\|_{L^2(\mathbb{T}\times[0,1])} \\ &\leq 10D^{1/2} = C' < \infty. \end{aligned}$$

By Theorem 7 we then see that

$$|\{(x,t) : |Th(x,t)| > \alpha\}| \leq \int_{t=0}^{1} \frac{c\,\|h(\cdot,t)\|_{L\log L(\mathbb{T})}}{\alpha} \leq \frac{c\cdot C'}{\alpha}.$$

This however is in contradiction to Equation 4, which holds for arbitrary large values of M. □

The proof of Theorem 8 now follows readily. Since T is of restricted weak type $(1,1)$ and of weak type $(2,2)$, its L^p bounds are on the order of magnitude of $\frac{1}{p-1}$ for p near 1. Hence by the Yano extrapolation theorem [21] we have that $\|Tf\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L\log L(\mathbb{T})}$, as desired.

The proof of the lemma above does suggest some problems of interest. For one thing, note that it relies on the compactness of $\mathbb{T}$ in inequality (5). This readily suggests the following.

Problem 1. *Suppose T is a sublinear translation invariant operator acting on measurable functions on $\mathbb{R}^n$ which is of restricted weak type $(1,1)$. At least when acting on simple functions, must T be of weak type $(2,2)$?*

In the above proof we showed that $\|h\|^2_{L^2(\mathbb{T}\times[0,1])} \leq \sum_{j=1}^N \|R_j f_j(x)\|^2_{L^2(\mathbb{T})}$. For $p > 2$ the inequality $\|h\|^2_{L^2(\mathbb{T}\times[0,1])} \leq \sum_{j=1}^N \|R_j f_j(x)\|^p_{L^p(\mathbb{T})}$ does not hold in general, and so we may not mimic the above proof to prove that a translation invariant operator acting on $L^1(\mathbb{T})$ of restricted weak type (1,1) is of weak type (p,p) for $p > 2$. Hence the following is open:

Problem 2. *Suppose T is a sublinear translation invariant operator acting on $L^1(\mathbb{T})$ which is of restricted weak type $(1,1)$. At least when acting on simple functions, must T be of weak type (p,p) for $1 < p < \infty$?*

Problem 2 also suggests the following generalization of Problem 1: Must a sublinear translation invariant operator T acting on measurable functions on $\mathbb{R}^n$ of restricted weak type $(1,1)$ be, at least when acting on simple functions, of weak type (p,p) for $1 < p < \infty$?

We now return back to the question of whether or not a sublinear translation invariant operator acting on $L^1(\mathbb{T})$ that is of restricted weak type (1,1) must be of weak type (1,1). It turns out that, as discovered Roger Jones and myself, the answer is no:

Theorem 9 ([5])**.** *There exists a sublinear translation invariant operator T acting on $L^1(\mathbb{T})$ which is of restricted weak type $(1,1)$ but not of weak type $(1,1)$.*

The operator T constructed above is a fun operator. It is sublinear, translation invariant, of restricted weak type (1,1), bounded on $L^p(\mathbb{T})$ for $1 < p \leq \infty$, maps $L \log L(\mathbb{T})$ into $L^1(\mathbb{T})$, but not of weak type (1,1).

This operator T cannot be readily modified to provide a counterexample to Conjecture 3. The existence of a sublinear translation invariant operator of restricted weak type (1,1) but not of weak type (1,1), however, should certainly cast doubt on whether Conjecture 3 holds. In this regard, though, it is interesting to note that the operator T constructed above is of the form

$$Tf(e^{i\theta}) = \sup_k |f * \mu_k(e^{i\theta})|,$$

where $\{\mu_k\}$ is a family of measures on $\mathbb{T}$ that can *not* be represented by functions in $L^1(\mathbb{T})$. It is worthwhile to contrast the structure of T with that of the operators considered in the following theorem of K. H. Moon:

Theorem 10 ([10])**.** *Let $\{g_n\}$ be a family of functions in $L^1(\mathbb{T})$, and let $Tf(e^{i\theta}) = \sup_n |f * g_n(e^{i\theta})|$. Then T is of weak type $(1,1)$ if and only if T is of restricted weak type $(1,1)$.*

Moon's positive result suggests a more viable way of proving Conjectures 1 and 2 than we have previously considered. In attempting to prove Conjecture 3 one is as a distinct disadvantage in that it encompasses operators with very little known structure, unlike the situation in Moon's theorem. Also, as is well known, the Dirichlet kernels $\{D_N(e^{i\theta})\}$ are not uniformly bounded in $L^1(\mathbb{T})$ but they are all integrable. Thus the Carleson Maximal operator is of the form considered in Moon's Theorem. Hence in order to prove Conjectures 1 and 2 it suffices to prove the following special case of Conjecture 3:

Conjecture 4. *Let $\{g_n\}$ be a family of functions in $L^1(\mathbb{T})$, and let $Tf(e^{i\theta}) = \sup_n |f * g_n(e^{i\theta})|$ be such that*

$$|\{e^{i\theta} : |T\chi_E(e^{i\theta})| > \alpha\}| \leq \frac{1}{\alpha} \|\chi_E\|_{L \log L(\mathbb{T})}$$

holds for any measurable set E in $\mathbb{T}$ and $\alpha > 0$. Then there exists a finite constant C such that

$$|\{e^{i\theta} : |Tf(e^{i\theta})| > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{L \log L(\mathbb{T})}$$

holds for any $f \in L \log L(\mathbb{T})$ and $\alpha > 0$.

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