RAMSEY-TYPE THEOREMS FOR SETS SATISFYING A GEOMETRIC REGULARITY CONDITION

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Abstract. We consider Ramsey-type problems associated to collections of sets in $\mathbb{R}^n$ satisfying a standard geometric regularity condition. In particular, let $\{R_j\}_{j=1}^N$ be a collection of measurable sets in $\mathbb{R}^n$ such that every R_j is contained in a cube Q_j whose sides are parallel to the axes and such that $|R_j|/|Q_j| \geq \rho > 0$. Moreover, suppose that there exists $0 < \gamma < \infty$ such that $|R_j|/|R_k| \leq \gamma$ for every j, k. We prove that there exists a subcollection of $\{R_j\}_{j=1}^N$ consisting of at least $R(N)$ sets that either have a point of common intersection or that are pairwise disjoint, where $R(N) \geq \left(\frac{N\rho}{(1+2\cdot\gamma^{1/n})^n}\right)^{1/2}$. If the sets in the collection $\{R_j\}$ are convex, we obtain the improved Ramsey estimate $R(N) \geq \left(3^{-n}\rho N\right)^{1/2}$. Applications of these results to weak type bounds of geometric maximal operators are provided.

1. Introduction

Ramsey's Theorem [3, 9] tells us that, given a collection of N sets $\{R_j\}_{j=1}^N$ in $\mathbb{R}^n$, we may find a subcollection $\{\tilde{R}_j\}_{j=1}^M$ of $\{R_j\}_{j=1}^N$ consisting of either pairwise disjoint or pairwise intersecting sets, where $M \geq \frac{1}{2}\log_2 N$. The order of magnitude of this bound is sharp, as was shown by Erdős in [2]. That being said, it is reasonable to expect an improved bound if the sets in $\{R_j\}$ are not completely random in nature but instead exhibit a particular type of structure. One example of such an improved bound was obtained by Larman, Matoušek, Pach, and Törőcsik in [7], who used Dilworth's Theorem [1] and arguments involving partially ordered sets to prove that if the sets in $\{R_j\}_{j=1}^N$ are convex, then one may find a subfamily of $N^{1/5}$ of them that are either pairwise disjoint or pairwise intersecting. In the same paper, they

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proved that if the sets $\{R_j\}_{j=1}^N$ are rectangular parallelepipeds in $\mathbb{R}^n$ whose sides are parallel to the axes, then one may find a subfamily of $C_n(\frac{N}{\log^{n-1} N})^{1/2}$ of them that are either pairwise disjoint or pairwise intersecting.

We consider here the question of the optimal Ramsey number associated to collections of sets in $\mathbb{R}^n$ satisfying a *geometric regularity condition* typically associated to problems arising in the theory of differentiation of integrals in harmonic analysis. Indeed, it would not be unreasonable to expect some relationship between the Ramsey number associated to a collection of sets $\mathcal{B} = \{R_j\}$ in $\mathbb{R}^n$ and the sharp weak type bound associated to the *maximal operator* $M_{\mathcal{B}}$ defined by

$$M_{\mathcal{B}}f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f|.$$

(For basic information regarding weak type bounds of geometric maximal operators, the reader is encouraged to consult [4] or [10].)

The most fundamental geometrical maximal operator in harmonic analysis is the so-called *Hardy-Littlewood maximal operator* M_{HL}, defined by

$$M_{HL}f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f| \ ,$$

where here the supremum is taken over all balls in $\mathbb{R}^n$. M_{HL} satisfies the *weak type* $(1,1)$ *inequality*

$$|\{x \in \mathbb{R}^n : M_{HL}f(x) > \alpha\}| \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f| = 3^n \int_{\mathbb{R}^n} \left| \frac{f}{\alpha} \right| \ .$$

The proof of this weak type inequality follows from the *Vitali Covering Theorem*, which states that, given a collection of balls $\{B_j\}_{j=1}^N$ in $\mathbb{R}^n$, there exists a disjoint subcollection $\{\tilde{B}_j\} \subset \{B_j\}$ such that

$$\left| \cup \tilde{B}_j \right| \geq 3^{-n} \left| \cup_{j=1}^N B_j \right| \ .$$

As observed by E. M. Stein in [10], a maximal operator $M_{\mathcal{B}}$ also satisfies a weak type $(1,1)$ condition if the sets in $\mathcal{B}$ satisfy the following regularity condition: there exists $0 < \rho < \infty$ such that, if $R \in \mathcal{B}$, there exists a cube Q containing R whose sides are parallel to the coordinate axes such that $\frac{|R|}{|Q|} \geq \rho$.

Now, a weak type $(1,1)$ condition on a geometric maximal operator $M_{\mathcal{B}}$ is the best we can hope for. Similarly, for a collection of sets $\{R_j\}$ in $\mathbb{R}^n$, the optimal Ramsey number associated to N sets is $N^{1/2}$. It is somewhat natural to conjecture whether, if a set $\mathcal{B}$ is associated to a maximal operator satisfying a weak type $(1,1)$ estimate, the associated

Ramsey number associated to N sets in the collection $\mathcal{B}$ is on the order of magnitude of $N^{1/2}$. We will see, although in general this is not the case, if $\mathcal{B}$ is a homothecy invariant collection of convex sets a weak $(1,1)$ bound on $M_{\mathcal{B}}$ will imply an optimal Ramsey estimate. This will be proven in part by showing that the Ramsey number associated to N convex sets satisfying the regularity condition above indeed is on the order of magnitude of $N^{1/2}$. This latter result is, to the best of our knowledge, originally due to Pach (see [8]), although we provide an independent proof here. We will provide a counterexample to the desired optimal Ramsey estimate associated to sets satisfying only the regularity condition indicated above, but on a more positive note prove an optimal Ramsey estimate for sets $\{R_j\}$ in $\mathbb{R}^n$ satisfying the above regularity condition with respect to ρ with the additional constraint that there exists $0 < \gamma < \infty$ such that $|R_j|/|R_k| \leq \gamma$ for all j, k.

2. Ramsey-Type Theorems for Convex Sets

Theorem 1. *Let $\{R_j\}_{j=1}^N$ be a collection of closed cubes in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes. Then we may find a subcollection $\{\tilde{R}_j\}$ of $\{R_j\}_{j=1}^N$ consisting of $R(N)$ sets that are either pairwise disjoint or that have a point of common intersection, where $R(N) \geq 2^{-n/2}N^{1/2}$.*

Proof. Motivated by the proof of the Vitali Covering Theorem, we reorder the sets in $\{R_j\}$ so that, without loss of generality, $|R_1| \leq |R_2| \leq \cdots \leq |R_N|$. (It is of interest that for our argument the sets R_j are arranged to be *increasing* in size, rather than decreasing as is the case with typical covering arguments.) We choose a subcollection $\{R'_j\}$ of $\{R_j\}$ as follows. Let $R'_1 = R_1$. Assuming now that $R'_1, \ldots, R'_j$ have been selected, let R'_{j+1} be the first R on the list $R_1, R_2, \ldots, R_N$ that is disjoint from $\cup_{k=1}^j R'_k$. If such an R does not exist, we terminate the selection process with R'_j.

Now, if the number of sets in $\{R'_j\}$ is at least as large as $2^{-n/2}N^{1/2}$, we simply set $\{\tilde{R}_j\} = \{R'_j\}$. Suppose this is not the case. Given $R \in \{R_j\}$, R must either be one of the R'_j or it must intersect a selected R'_j whose size is at most that of R (here we take advantage of the initial ordering of the R_j.) Note that in the latter case we moreover have that R must intersect one of the 2^n corners of a selected R'_j. As there are less than $2^n \cdot 2^{-n/2}N^{1/2} = 2^{n/2}N^{1/2}$ points in $\mathbb{R}^n$ that are corners of the R'_j, we have by the pigeonhole principle that one of the corners must be contained in more than $\frac{N}{2^{n/2}N^{1/2}} = 2^{-n/2}N^{1/2}$ sets in $\{R_j\}$. Letting

$\{\tilde{R}_j\}$ be the sets in $\{R_j\}$ that intersect this corner, we see the desired result holds. $\square$

We now consider an analogue of the above theorem where we replace cubes by convex sets in $\mathbb{R}^n$. For general collections of convex sets, we cannot obtain a Ramsey number on the order of magnitude of $N^{1/2}$. In fact, by considering the Ramsey numbers associated to special arrangements of line segments in the plane Larman et al. showed in [7] that the associated $R(N)$ is smaller than $N^{0.431}$, and the more recent result of Kynčl [6] indicates that we can do no better than $N^{\log 8/\log 169} \leq N^{0.406}$. We are, however, able to obtain a Ramsey estimate on the order of magnitude of $N^{1/2}$ by requiring that the convex sets under consideration satisfy the regularity condition that there exists $0 < \rho < \infty$ such that, if S is one of the convex sets, there exists a cube Q containing S whose sides are parallel to the axes and such that $|S|/|Q| \geq \rho$. In particular, we obtain the following.

Theorem 2. *Let $\{R_j\}_{j=1}^N$ be a collection of convex sets in $\mathbb{R}^n$ such that, given $R \in \{R_j\}$, there exists a cube Q in $\mathbb{R}^n$ whose sides are parallel to the axes such that $R \subset Q$ and such that $|R|/|Q| \geq \rho > 0$. Then we may find a subcollection $\{\tilde{R}_j\}$ of $\{R_j\}_{j=1}^N$ consisting of $R(N)$ sets that are either pairwise disjoint or that have a point of common intersection, where $R(N) \geq (3^{-n}\rho N)^{1/2}$.*

Proof. Without loss of generality, we assume $|R_1| \leq |R_2| \leq \cdots \leq |R_N|$. We choose a subcollection $\{R'_j\}$ of $\{R_j\}$ as we did in the previous theorem; in particular we set $R'_1 = R_1$ and, assuming $R'_1, \ldots, R'_j$ have been chosen, let R'_{j+1} be the first set on the list $R_1, \ldots, R_N$ that is disjoint from $\cup_{k=1}^{j} R'_k$. If such an R does not exist, we terminate the selection process with R'_j.

If the number of sets in $\{R'_j\}$ is at least as large as $(3^{-n}\rho N)^{1/2}$, we simply set $\{\tilde{R}_j\} = \{R'_j\}$. Suppose this is not the case. Given $R \in \{R_j\}$, R must either be one of the R'_j or it must intersect a selected R'_j whose size is at most that of R. By the pigeonhole principle, there must be at least one selected R'_j (we call it R') that relates to more than $N/(3^{-n}\rho N)^{1/2} = (3^n\rho^{-1}N)^{1/2}$ sets in $\{R_j\}$ of size at least that of R'. We label those sets as $S_1, \ldots, S_k$, where $k > (3^n\rho^{-1}N)^{1/2}$. Now, each S_j is contained in a cube Q_j whose sides are parallel to the axes and such that $|S_j|/|Q_j| \geq \rho$. Moreover, R' is contained in a cube Q' whose sides are parallel to the axes, where $|R'|/|Q'| \geq \rho$. Blowing up cubes, without loss of generality, we may assume $|S_j|/|Q_j| = |R'|/|Q'| = \rho$. Let $3Q'$ denote the cube in $\mathbb{R}^n$ with sides parallel to the axes with the same center as that of Q' and 3^n times the volume. *Since each S_j is convex,* there exists a homothecy $T_j \subset S_j$ of S_j with $T_j \subset 3Q'$, which results from shrinking Q_j down to size $|Q'|$. In particular, $|T_j|/|3Q'| = 3^{-n}|T_j|/|Q'| = 3^{-n}\rho$. As there are more than $(3^n\rho^{-1}N)^{1/2}$ of these homothecies, there is a point in $3Q'$ contained in more than

$$\frac{(3^n\rho^{-1}N)^{1/2} \cdot 3^{-n}\rho|3Q'|}{|3Q'|} = \left(3^{-n}\rho N\right)^{1/2}$$

of the T_j. The desired result holds. $\square$

3. Ramsey-Type Theorems for sets satisfying a geometric regularity condition

We now seek an analogue of the above two theorems where the sets under consideration do not have a convexity condition but retain the regularity condition that there exists $0 < \rho < \infty$ such that, if S is one of the sets, there exists a cube Q containing S whose sides are parallel to the axes and such that $|S|/|Q| \geq \rho$. Unfortunately, this constraint alone provides no meaningful positive result. For example, we could consider a collection of sets $\{R_j\}_{j=1}^N$ in the plane where $R_j = (S_j \cup (2^j, 2^{j+1})) \times [0, 2^{j+1}]$ for arbitrary sets $S_j \subset [0, 1]$. The associated ρ for this collection is $1/2$, but as the S_j are effectively random sets in $[0, 1]$ we may obtain no nontrivial estimate on the Ramsey numbers associated to $\{R_j\}_{j=1}^N$.

Notice, however, that in the above example the R_j's are growing in size, and it is natural to suppose that we could rescue the situation by requiring a uniform bound on $|R_j|/|R_k|$. These suppositions are validated by the following:

Theorem 3. *Let $\{R_j\}_{j=1}^N$ be a collection of measurable sets in $\mathbb{R}^n$ such that every R_j is contained in a cube Q_j whose sides are parallel to the axes and such that $|R_j|/|Q_j| \geq \rho > 0$. Moreover, suppose that there exists $0 < \gamma < \infty$ such that $|R_j|/|R_k| \leq \gamma$ for every j, k. Then we may find a subcollection $\{\tilde{R}_j\}$ of $\{R_j\}_{j=1}^N$ consisting of $R(N)$ sets that are either pairwise disjoint or that have a point of common intersection, where $R(N) \geq \left(\frac{N\rho}{(1+2\cdot\gamma^{1/n})^n}\right)^{1/2}$.*

Proof. As we have done in the previous two theorems, we assume without loss of generality that $|R_1| \leq |R_2| \leq \cdots \leq |R_N|$. We choose a subcollection $\{R'_j\}$ of $\{R_j\}$ as follows. Let $R'_1 = R_1$. Assuming now that $R'_1, \ldots, R'_j$ have been selected, let R'_{j+1} be the first R on the list $R_1, R_2, \ldots, R_N$ that is disjoint from $\cup_{k=1}^j R'_k$. If such an R does not exist, we terminate the selection process with R'_j.

Let $x_{\gamma,\rho} = \left(\frac{N\rho}{(1+2\cdot\gamma^{1/n})^n}\right)^{1/2}$. If the number of sets in $\{R'_j\}$ is at least as large as $x_{\gamma,\rho}$, we simply set $\{\tilde{R}_j\} = \{R'_j\}$. Suppose this is not the case. Given $R \in \{R_j\}$, R must either be one of the R'_j or it must intersect a selected R'_j whose size is at most that of R. By the pigeonhole principle, there must be at least one selected R'_j (we call it R') that relates to more than $N/x_{\gamma,\rho}$ sets in $\{R_j\}$ of size at least that of R'. We label those sets as $S_1, \ldots, S_k$, where $k > N/x_{\gamma,\rho}$. Now, each S_j is contained in a cube Q_j whose sides are parallel to the axes and such that $|S_j|/|Q_j| \geq \rho$. Moreover, R' is contained in a cube Q' whose sides are parallel to the axes, where $|R'|/|Q'| \geq \rho$. Blowing up cubes, without loss of generality, we may assume $|S_j|/|Q_j| = |R'|/|Q'| = \rho$. Since by hypothesis $|S_j|/|R'| \leq \gamma$, we see that for every j, $|Q_j| = \rho^{-1}|S_j| \leq \frac{\gamma}{\rho}|R'| = \gamma|Q'|$. As Q_j either intersects or coincides with Q' we then have $S_j \subset Q_j \subset (1 + 2\cdot\gamma^{1/n})Q'$ where here cQ' denotes the cube in $\mathbb{R}^n$ of volume $c^n|Q'|$ with sides parallel to the axes and sharing the same center as Q'. By the pigeonhole principle, we recognize there must be a point in $(1 + 2\cdot\gamma^{1/n})Q'$ contained in more than

$$\frac{\frac{N}{x_{\gamma,\rho}}|R'|}{(1 + 2\cdot\gamma^{1/n})^n|Q'|} = \frac{\frac{N}{x_{\gamma,\rho}}\rho}{(1 + 2\cdot\gamma^{1/n})^n} = x_{\gamma,\rho}$$

of the S_j. The desired result holds. □

4. Ramsey Estimates, Geometric Maximal Operators, and Future Directions

This foray into establishing elementary Ramsey-type estimates associated to sets in $\mathbb{R}^n$ has been both informative and promising. Indeed, we have near optimal ($\sim N^{1/2}$) estimates for convex sets also satisfying a regularity condition. Convexity is crucial for this: at one stage of the proof of Theorem 2 we needed the fact that if a convex set S intersects a cube Q, there is a homothecy $T \subset S$ of S that is contained in a concentric 3-fold dilate of Q. With loss of convexity the near optimal Ramsey estimate breaks down as was indicated at the beginning of the third section. It is of interest that the basis $\mathcal{B}$ associated to the sets R_j in this example is such that $M_{\mathcal{B}}$ is of weak type $(1,1)$, this is essentially proven by E. M. Stein in Chapter I of [10]. Hence if $\{R_j\}_{j=1}^N \subset \mathcal{B}$ and $M_{\mathcal{B}}$ is of weak type $(1,1)$, we do not necessarily have that $R(N) \sim N^{1/2}$. We are tempted to conjecture, however, that if $\mathcal{B}$ consists only of convex sets we do obtain an optimal Ramsey estimate:

Conjecture 1. *Let $\mathcal{B}$ be a collection of convex sets in $\mathbb{R}^n$ such that the associated geometric maximal operator $M_{\mathcal{B}}$ is of weak type $(1,1)$.*

Then there exists a constant C such that, given $\{R_j\}_{j=1}^N \subset \mathcal{B}$, there exists a subset $\{\tilde{R}_j\} \subset \{R_j\}_{j=1}^N$ consisting of $CN^{1/2}$ sets that are either pairwise disjoint or that pairwise intersect.

Note that we cannot replace "pairwise intersect" with "have a point of common intersection". It is quite easy to construct an arbitrarily large number of convex sets (approximating line segments in the plane) that pairwise intersect but never have more than two overlap at any given point. The associate maximal operator has a weak type $(1,1)$ bound of 2, but we get no meaningful Ramsey estimate involving a condition regarding points of common intersection of sets.

We are able to show that the above conjecture holds in the important case that $\mathcal{B}$ is a *homothecy invariant* collection of convex sets.

Theorem 4. *Let $\mathcal{B}$ be a homothecy invariant collection of convex sets in $\mathbb{R}^n$ such that the associated geometric maximal operator $M_{\mathcal{B}}$ is of weak type $(1,1)$. Then there exists a constant $C > 0$ such that, given $\{R_j\}_{j=1}^N \subset \mathcal{B}$, there exists a subset $\{\tilde{R}_j\} \subset \{R_j\}_{j=1}^N$ consisting of $CN^{1/2}$ sets that are either pairwise disjoint or that have a point of common intersection.*

Proof. R. Moriyón proved in [4, Appendix III] that if $\mathcal{B}$ is a homothecy invariant collection of convex sets, the maximal operator $M_{\mathcal{B}}$ is of weak type $(1,1)$ if and only if the sets in $\mathcal{B}$ satisfy the regularity condition we have considered here; i.e. that each $R_j \in \mathcal{B}$ is contained in a cube Q_j whose sides are parallel to the axes and such that $|R_j|/|Q_j| \geq \rho > 0$. The proof then follows from Theorem 2. $\square$

We could conversely ask if an optimal Ramsey estimate for subcollections of a homothecy invariant basis $\mathcal{B}$ implies that the associated maximal operator $M_{\mathcal{B}}$ is of weak type $(1,1)$. It is unclear what role convexity plays here.

Conjecture 2. *Let $\mathcal{B}$ be a homothecy invariant collection of (convex) sets in $\mathbb{R}^n$ such that for some constant $C > 0$, given $\{R_j\}_{j=1}^N \subset \mathcal{B}$, there exists a subset $\{\tilde{R}_j\} \subset \{R_j\}_{j=1}^N$ consisting of $CN^{1/2}$ sets that are either pairwise disjoint or that have a point of common intersection. Then $M_{\mathcal{B}}$ is of weak type $(1,1)$.*

We remark that the condition of homothecy invariance above cannot be dispensed with. For example, if $\mathcal{B}$ consisted of all the rectangular parallelepipeds in $\mathbb{R}^n$ containing the origin, we would have an optimal Ramsey estimate but the maximal operator $M_{\mathcal{B}}$ would not be of weak type $(1,1)$.

Also of particular interest is the notion that *non-optimal ($\not\sim N^{1/2}$) but sharp* Ramsey estimates might be encoded in non-weak type $(1,1)$ but nonetheless sharp weak type bounds for certain maximal operators. As an example, the *strong maximal operator* M_S is defined by

$$M_S f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f| \ ,$$

where the supremum is taken over rectangular parallelepipeds in $\mathbb{R}^n$ whose sides are parallel to the axes. By a theorem of Jessen, Marcinkiewicz, and Zygmund [5] we have that

$$|\{x \in \mathbb{R}^n : M_S f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \left|\frac{f}{\alpha}\right| \left(1 + \left(\log^+ \left|\frac{f}{\alpha}\right|\right)^{n-1}\right)$$

where the exponent $n-1$ is sharp. Possibly only a coincidence is in play, but Larman et al. showed in [7] that, given N rectangular parallelepipeds in $\mathbb{R}^n$ whose sides are parallel to the axes, there exist $C_n \left(\frac{N}{(\log N)^{n-1}}\right)^{1/2}$ parallelepipeds in that collection that are either pairwise disjoint or that pairwise intersect. Although in their paper they suggest that an optimal $C_n N^{1/2}$ Ramsey estimate is reasonable to expect, it is a tantalizing notion that the estimate they obtain is optimal and is in some sense encoded in the sharp weak type bound for the strong maximal operator. We formalize these observations with the following conjectures:

Conjecture 3. *Suppose there exists a convex increasing function $\Phi(x) : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and a homothecy invariant collection of convex sets $\mathcal{B}$ in $\mathbb{R}^n$ such that*

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \alpha\}| \leq \int_{\mathbb{R}^n} \Phi\left(\left|\frac{f}{\alpha}\right|\right) \ .$$

Then there exists a constant $C_{\mathcal{B}} > 0$ such that, given a collection $\{R_j\}_{j=1}^N \subset \mathcal{B}$, there exists a subcollection of $C_{\mathcal{B}}(\frac{N^2}{\Phi(N)})^{1/2}$ sets that are either pairwise disjoint or that pairwise intersect.

Conversely,

Conjecture 4. *Suppose there exists a convex increasing function $\Phi(x) : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and a homothecy invariant collection of convex sets $\mathcal{B}$ in $\mathbb{R}^n$ such that, given a collection $\{R_j\}_{j=1}^N \subset \mathcal{B}$, there exists a subcollection of $(\frac{N^2}{\Phi(N)})^{1/2}$ sets that are either pairwise disjoint or that have a point of common intersection. Then*

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \Phi\left(\left|\frac{f}{\alpha}\right|\right) \ .$$

Another point of consideration is whether we can weaken "have a point of common intersection" to "pairwise intersect" in Conjectures 2 and 4. These questions and conjectures are the subject of ongoing research.

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