ON A THEOREM OF BESICOVITCH AND A PROBLEM IN ERGODIC THEORY

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Abstract. In 1935, Besicovitch proved a remarkable theorem indicating that an integrable function f on $\mathbb{R}^2$ is strongly differentiable if and only if its associated strong maximal function $M_S f$ is finite a.e. We consider analogues of Besicovitch's result in the context of ergodic theory, in particular discussing the problem of whether or not, given a (not necessarily integrable) measurable function f on a nonatomic probability space and a measure preserving transformation T on that space, the ergodic averages of f with respect to T converge a.e. if and only if the associated ergodic maximal function $T^* f$ is finite a.e. Of particular relevance to this discussion will be recent results in the field of inhomogeneous diophantine approximation.

Let f be an integrable function on $\mathbb{R}^2$. A classical result in analysis, the *Lebesgue Differentiation Theorem*, tells us that, for a.e. $x \in \mathbb{R}^2$, the averages of f over disks shrinking to x tend to $f(x)$ itself. More precisely, we have that

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f = f(x) \quad \text{a.e.},$$

where $B(x,r)$ denotes the open disk centered at x of radius r and $|B(x,r)| = \pi r^2$ denotes the area of the disk. For a proof of this result, the reader is encouraged to consult [9].

What happens if we average over sets other than disks, say, open rectangles? It turns out that there exist integrable functions f on $\mathbb{R}^2$ such that, for a.e. $x \in \mathbb{R}^2$, there exists a sequence of rectangles $\{R_{x,j}\}$ shrinking toward x for which

$$\lim_{j \to \infty} \frac{1}{|R_{x,j}|} \int_{R_{x,j}} f$$

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fails to converge. The news gets even more interesting. In fact, one can construct a function $f = \chi_E$ (that is *the characteristic function of a set* $E \subset \mathbb{R}^2$) such that, for a.e. $x \in \mathbb{R}^2$, there exists a sequence of rectangles $\{R_{x,j}\}$ shrinking toward x for which

$$\lim_{j\to\infty} \frac{1}{|R_{x,j}|} \int_{R_{x,j}} \chi_E$$

fails to converge. (See [3] for a nice exposition of this result. This result is closely related to the well-known *Kakeya Needle Problem*, and the interested reader is highly encouraged to consult [2] for more information on this topic.)

If we restrict the class of rectangles that we allow ourselves to average over, we obtain better results. In [5], Jessen, Marcinkiewicz, and Zygmund proved that if $\mathcal{B}_2$ consists of all the open rectangles in $\mathbb{R}^2$ *whose sides are parallel to the coordinate axes*, then for any function $f \in L^p(\mathbb{R}^2)$ with $1 < p \leq \infty$ one has

$$\lim_{j\to\infty} \frac{1}{|R_j|} \int_{R_j} f \;=\; f(x)$$

for a.e. $x \in \mathbb{R}^2$, where here $\{R_j\}$ is any sequence of rectangles in $\mathcal{B}_2$ shrinking toward x. (In this scenario we would say f is *strongly differentiable*.) Jessen, Marcinkiewicz, and Zygmund proved this by showing that the *strong maximal operator* M_S, defined by

$$M_S f(x) = \sup_{x\in R\in\mathcal{B}_2} \frac{1}{|R|} \int_R |f|$$

satisfies for every $1 < p < \infty$ the *weak type* (p,p) estimate

$$\left|\{x \in \mathbb{R}^2 : M_S f(x) > \alpha\}\right| \leq C_p \left(\frac{\|f\|_{L^p}}{\alpha}\right)^p .$$

This illustrates a paradigm that has been highly successful in the theory of differentiation of integrals. Namely, suppose one is given a collection of open sets $\mathcal{B} \subset \mathbb{R}^n$ and one wishes to ascertain whether, given a function f on $\mathbb{R}^n$, for a.e. x one must have

$$\lim_{j\to\infty} \frac{1}{|S_j|} \int_{S_j} f = f(x) \tag{0.1}$$

whenever $\{S_j\}$ is a sequence of sets in $\mathcal{B}$ shrinking toward x. (Here we assume that every point x is contained in a set in $\mathcal{B}$ of arbitrarily small diameter.) We may associate to the collection $\mathcal{B}$ a *maximal operator* $M_{\mathcal{B}}$ defined by

$$M_{\mathcal{B}} f(x) = \sup_{x\in S\in\mathcal{B}} \frac{1}{|S|} \int_S |f| .$$

It turns out that (0.1) will hold for every f in $L^p(\mathbb{R}^n)$ a.e. provided $M_{\mathcal{B}}$ satisfies a weak type (p,p) estimate. A deep theorem of E. M. Stein [8] tells us that, provided $\mathcal{B}$ is translation invariant in the sense that, if $S \in \mathcal{B}$ then every translate of S also lies in $\mathcal{B}$, the above limits will hold for every $f \in L^p(\mathbb{R}^n)$ *only if* $M_{\mathcal{B}}$ satisfies a weak type (p,p) estimate. It is for this reason that maximal operators are an indispensable tool for mathematicians working with the topic of differentiation of integrals.

Having said that, it is interesting to consider the paper in *Fundamenta Mathematicae* immediately preceding the famous paper of Jessen, Marcinkiewicz, and Zygmund. In this paper [1], *On differentiation of Lebesgue double integrals*, Besicovitch proved that, given any integrable function f on $\mathbb{R}^2$, if $M_S f$ is finite a.e., then for a.e. x we have

$$\lim_{j\to\infty} \frac{1}{|R_j|} \int_{R_j} f = f(x)$$

whenever $\{R_j\}$ is a sequence of sets in $\mathcal{B}_2$ shrinking to x. Of course, if $f \in L^p(\mathbb{R}^2)$ for $1 < p < \infty$, the quantitative weak type (p,p) bound satisfied by M_S implies that $M_S f$ will be finite a.e. It is for this reason that this paper of Besicovitch has received comparatively little attention. However, it is of note that Besicovitch provides a mechanism for obtaining a.e. differentiability results bypassing the need for quantitative weak type bounds on an associated maximal operator.

Let us provide an illustration of the usefulness of this approach. Let $f(x,y) = g(x)\chi_{[0,1]\times[0,1]}(x,y)$ be a function on $\mathbb{R}^2$, where $g \in L^1(\mathbb{R})$. Note $f \in L^1(\mathbb{R}^2)$ but not necessarily in $L^p(\mathbb{R}^2)$ for any $p > 1$. Suppose we wish to show that f is strongly differentiable. We can use the Fubini theorem combined with the weak type $(1,1)$ bounds of the Hardy-Littlewood maximal operator to show that $M_S f$ is finite a.e., so by the Besicovitch Theorem we know that f is strongly differentiable. However, M_S is *not* of weak type $(1,1)$. Note that here we did not show that *every* function in $L^1(\mathbb{R}^2)$ is strongly differentiable, only that *some* of these functions are.

Many results in the study of differentiation of integrals have a "companion" result in ergodic theory; for instance the Lebesgue Differentiation Theorem is structurally very similar to that of the *Birkhoff Ergodic Theorem* on integrable functions. This observation may be found at least as far back as the work of Wiener [10]. In that regard, it is natural to consider what the companion result of Besicovitch's Theorem might be, when replacing the strong maximal operator M_S by an ergodic maximal operator. We are led immediately to the following conjecture.

Conjecture 1. *Let T be a measure preserving transformation on the nonatomic probability space (X, Σ, μ) and let f be a μ-measurable function on that space. If $T^* f(x)$ is finite μ-a.e., where $T^* f$ is the ergodic maximal function defined by*

$$T^* f(x) = \sup_{N \geq 1} \frac{1}{N} \left| \sum_{j=0}^{N-1} f(T^j x) \right| ,$$

then the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

exists μ-a.e.

We remark that if f is integrable, then by the Birkhoff Ergodic Theorem the above limit automatically exists. If $f = f^+ - f^-$ as difference of nonnegative measurable functions f^+ and f^-, then by the proof of the Birkhoff Ergodic Theorem the above limit still holds provided that *at least one* of the functions f^+ and f^- is integrable. (The reader may consult [7] for a proof of Birkhoff's classical result verifying that Fatou's Lemma easily extends the given argument to the more general situation.) Thus, the interesting case is where $f = f^+ - f^-$ with

$$\int_X f^+ d\mu = \int_X f^- d\mu = \infty \; . \tag{0.2}$$

The main purpose of this note, aside from advertising the above conjecture, is to consider what happens when T corresponds to an ergodic transformation associated to an irrational rotation on $[0, 1)$ (identified with the unit circle $\mathbb{T}$), and $f(x) = \frac{1}{x - 1/2}$. This scenario is so natural to consider that the reader might be surprised to find that it has not been treated before. (At least, the authors are unaware of any explicit treatment of this example.) In considering this situation, several issues immediately come to mind. First of all, f clearly satisfies (0.2), so we are not in a situation where we can apply the ergodic theorem. However, f exhibits a natural cancellation, so one might wonder whether the ergodic averages of f tend to 0 a.e. And, moreover, even if the ergodic averages of f did not tend to 0 a.e., it is still possible that the ergodic maximal function $T^* f$ is finite a.e. In that regard, this example seems to be a very worthy candidate for a counterexample of Conjecture 1.

It turns out that neither the ergodic averages of f with respect to T converge a.e. nor is the ergodic maximal function T^*f finite a.e. The proof of the former follows readily from a theorem of Khintchine on the topic of inhomogeneous Diophantine approximation. The proof of the latter is much more subtle, following from relatively recent results of Kim [6].

Theorem 1. *Let ξ be an irrational number, and define the measure preserving transformation T on $[0,1)$ by*

$$Tx = (x+\xi) \bmod 1 .$$

Define the function f on $[0,1)$ by

$$f(x) = \frac{1}{x-1/2} .$$

If $x \in [0,1)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

fails to converge to a finite number. Moreover for a.e. $x \in [0,1)$ we have

$$T^*f(x) = \sup_{N\geq 1} \frac{1}{N} \left| \sum_{j=0}^{N-1} f(T^j x) \right| = \infty .$$

Proof. We first show that at no point $x \in [0,1)$ does

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

converge to a finite value. It will be convenient for us to use the notation

$$\|x\| = \min_{n\in\mathbb{Z}} |x-n| .$$

We proceed by contradiction. Suppose for a given $x \in [0,1)$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) = L < \infty .$$

Note that since ξ is an irrational number, by a theorem of Khintchine on inhomogeneous Diophantine approximation (see [4], p. 267) we have

$$\left\| q\xi + x - \frac{1}{2} \right\| < \frac{1}{q} ,$$

and thus

$$|f(T^q x)| = |f((q\xi + x) \mod 1)| > q$$

for infinitely many positive integers q. Observe that

$$\frac{1}{q+1}\sum_{j=0}^{q} f(T^j x) - \frac{1}{q}\sum_{j=0}^{q-1} f(T^j x)$$

$$= \frac{q}{q+1}\cdot\frac{1}{q}f(T^q x) - \frac{1}{q+1}\cdot\frac{1}{q}\sum_{j=0}^{q-1} f(T^j x)\ .$$

Hence

$$\begin{aligned}
&\limsup_{q\to\infty}\left|\frac{1}{q+1}\sum_{j=0}^{q} f(T^j x) - \frac{1}{q}\sum_{j=0}^{q-1} f(T^j x)\right| \\
= \ &\limsup_{q\to\infty}\left|\frac{q}{q+1}\cdot\frac{1}{q}\, f(T^q x) - \frac{1}{q+1}\cdot\frac{1}{q}\sum_{j=0}^{q-1} f(T^j x)\right| \\
= \ &\limsup_{q\to\infty}\frac{1}{q}\,|f(T^q x)| \geq 1\ .
\end{aligned}$$

Accordingly, $\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} f(T^j x)$ cannot converge to a finite value L, contradicting the supposition that these ergodic averages indeed did converge to L.

We now show that for a.e. $x \in [0,1)$ we have

$$T^* f(x) = \sup_{N\geq 1}\frac{1}{N}\left|\sum_{j=0}^{N-1} f(T^j x)\right| = \infty\ .$$

To show this we use the relatively recent remarkable result of D. H. Kim [6] that

$$\liminf_{q\to\infty} q\cdot\left\|q\xi + x - \frac{1}{2}\right\| = 0$$

for a.e. $x \in [0,1)$. Thus

$$\limsup_{q\to\infty}\frac{1}{q}\,|f(T^q x)| = \infty$$

for a.e. $x \in [0,1)$. Let $x \in [0,1)$ be such that the above limit superior is infinite. We show that $T^* f(x) = \infty$. Again we proceed by contradiction. Suppose that

$$\sup_{N\geq 1}\frac{1}{N}\left|\sum_{j=0}^{N-1} f(T^j x)\right| = M < \infty.$$

Then, repeating the above calculation, we have the contradiction

$$\begin{aligned} 2M &\geq \limsup_{q\to\infty} \left| \frac{1}{q+1} \sum_{j=0}^{q} f(T^j x) - \frac{1}{q} \sum_{j=0}^{q-1} f(T^j x) \right| \\ &= \limsup_{q\to\infty} \frac{1}{q} \left| f(T^q x) \right| = \infty \ . \end{aligned}$$

□

In addition to Conjecture 1, we wish to indicate another conjecture the reader might find of interest. In Theorem 1 we showed that for a.e. $x \in [0,1)$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

does not converge to a finite number. What type of divergence is exhibited? By the apparent symmetries involved we would ordinarily expect that the ergodic averages of f do not converge to either positive or negative infinity, noting that the set

$$S = \left\{ x \in [0,1) : \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) = \infty \right\}$$

is invariant under the ergodic transformation T and thus either of measure 0 or 1, and it would be strange for these averages to converge to $+\infty$ a.e. but not $-\infty$. Nonetheless, we do not have a proof of this, and the issue appears hard as the techniques involved in Kim's result do not indicate, given $x \in [0,1)$, on which "side" of $1/2$ the points $(x + q\xi) \mod 1$ close to $1/2$ lie. We formalize these ideas in the following:

Conjecture 2. *Let ξ be an irrational number, and define the measure preserving transformation T on $[0,1)$ by*

$$Tx = (x + \xi) \text{ mod } 1 \ .$$

Define the function f on $[0,1)$ by

$$f(x) = \frac{1}{x - 1/2} \ .$$

Then, for a.e. $x \in [0,1)$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) = \infty$$

and

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) = -\infty\ .$$

Conjectures 1 and 2 are topics of ongoing research.

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