A Discrete Analog of Quantum Unique Ergodicity on Circulant Graphs

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Quantum ergodicity

Theorem 1 (Schnirelman, Zelditch, Colin de Verdière)

For ergodic dynamics and $\{\psi_{j_k}\}$ an eigenfunction sequence with density one,

$$\int_\Omega |\psi_{j_k}|^2 f \, dA \to \frac{1}{\mathrm{Vol}(\Omega)} \cdot \int_\Omega f \, dA$$

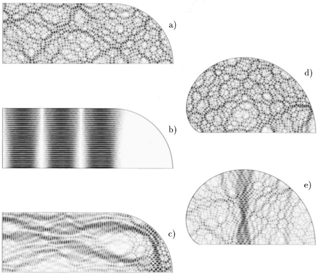

Figure 1: Image Bäcker, Schubert and Stifter '98

Quantum Unique Ergodicity (QUE)

$$\int_\Omega |\psi_{j_k}|^2 f\, dA \to \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega f\, dA \text{ for every sequence } \{\psi_{j_k}\}.$$

- QE on billiards, Zelditch-Zworski '96.
- Conjecture: QUE holds on compact negatively curved manifolds, Rudnick-Sarnak.
- QUE on arithmetic surfaces, Lindenstrauss '06.
- Compact negatively curved manifolds are at least half delocalized, Anantharaman-Nonnenmacher '07.
- Ergodic billiards are not quantum unique ergodic, Hassell '10.
- No QE for quantum star graphs, Berkolaiko-Keating-Winn '04.
- QE for expanding quantum graphs, Anantharaman-Ingremeau-Sabri-Winn '21.

Spectral graph theory

- *Graph* $G = (V, E)$ a set of vertices V connected by edges E.
- $(i, j) \in E$ then vertices i and j are *adjacent* $i \sim j$.
- *Adjacency matrix* $A(G)$ is a $|V| \times |V|$ matrix with $[A(G)]_{ij} = 1$ if $i \sim j$ and 0 otherwise.
- *Semiclassical limit* is limit of sequence of graphs of increasing size.

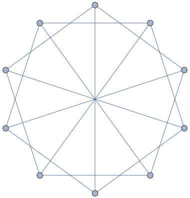

Figure 2: Circulant graph $C_{10}(2, 5)$.

A discrete version of quantum unique ergodicity

- Let $f \in \ell^2(V_n)$ with $||f||_{\ell^2} = 1$.
- A *quantum probability measure* associated to f is,

$$\mu_f = \sum_{v \in V_n} |f(v)|^2 \delta_v \ . \tag{1}$$

- Let $\{G_n = (V_n, E_n)\}_{n \in \mathcal{I}}$ be a sequence of graphs, $\mathcal{I} \subseteq \mathbb{N}$.
- Let $U_n \subset V_n$ with $\lim_{n\to\infty} \frac{|U_n|}{n} = P$.
- Let f_n be an eigenfunction of the adjacency matrix $A_n(G_n)$ with $||f_n||_{\ell^2} = 1$.

For $\{f_n\}_{n \in \mathcal{I}}$ and $\{U_n\}_{n \in \mathcal{I}}$ we say Discrete Quantum Unique Ergodicity (DQUE) holds if,

$$\lim_{n\to\infty} \mu_{f_n}(U_n) = P \ . \tag{2}$$

Cayley graphs of quasirandom groups

- DQUE definition, Magee-Thomas-Zhao '23.
- A finite group G is *$\mathcal{D}(G)$-quasirandom* if the minimum dimension of a non-trivial irreducible representation is $\mathcal{D}(G)$.

Theorem 2 (Magee-Thomas-Zhao '23)

For Cayley graphs that are $\log^2(|G|)$-quasirandom $\exists$ orthonormal basis $\mathcal{B}_n$ of eigenfunctions of $A(G_n)$ s.t. all sequences of eigenfunctions in $\mathcal{B}_n$ have DQUE.

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- Full results for Cayley graphs of quasirandom groups are more general.
- They also prove the same results for orthonormal bases of real valued eigenfunctions.

Circulant graphs

- Cayley graph of $\mathbb{Z}_n$.
- Vertices $\{1, \ldots, n\}$.
- Let $\vec{a} = (a_1, \ldots, a_d)$, s.t. $0 < a_1 < a_2 < \cdots < a_d < n/2$.
- Circulant graph $C_n(\vec{a})$: Edge $(i, j) \in E$ iff $|i - j| \equiv a_h \mod n$.
- $C_n(\vec{a})$ connected iff $\gcd(a_1, \ldots, a_d, n) = 1$.

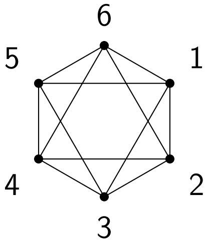

Figure 3: Circulant graph $C_6(1, 2)$.

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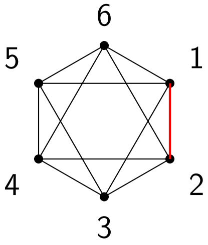

Figure 4: Circulant graph $C_6(1, 2)$.

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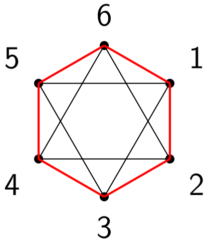

Figure 5: Circulant graph $C_6(1, 2)$.

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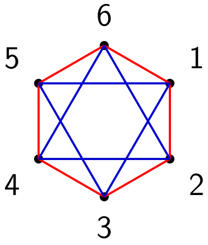

Figure 6: Circulant graph $C_6(1, 2)$.

Circulant matrix

$$\begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}$$

Adjacency matrix of circulant graph is circulant matrix.

$$A(C_6(1,2)) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{3}$$

DQUE for circulant graphs

Proposition 3 (H.-Pruss)

For circulant graphs $\{C_p(\vec{a})\}$ with p prime $\exists$ orthonormal bases $\mathcal{B}_n$ of eigenfunctions of $A(C_p(\vec{a}))$ s.t. all sequences of eigenfunctions in $\mathcal{B}_n$ have DQUE.

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- p prime: $C_p(\vec{a})$ connected and eigenvalues max degeneracy 2.
- Irreducible representations of $\mathbb{Z}_n$ 1-dimensional.
- DQUE follows from eigenvectors of circulant matrices,

$$\vec{v}_j = \frac{1}{\sqrt{n}}(1, \omega^j, \omega^{2j}, ..., \omega^{(n-1)j})^T \tag{4}$$

with $\omega = e^{2\pi i/n}$.

Real bases for circulant graphs

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- For circulant graphs $\lambda_j = \lambda_{n-j}$.
- Real orthonormal basis $\mathcal{B}_n = \{\vec{v}_0, \vec{c}_1, \vec{s}_1, \dots, \vec{c}_{\frac{n-1}{2}}, \vec{s}_{\frac{n-1}{2}}\}$,

$$\vec{c}_j = \sqrt{\frac{2}{n}} \left(1, \cos\left(\frac{2\pi j}{n}\right), \cos\left(\frac{4\pi j}{n}\right), \dots, \cos\left(\frac{2(n-1)\pi j}{n}\right)\right)$$

$$\vec{s}_j = \sqrt{\frac{2}{n}} \left(0, \sin\left(\frac{2\pi j}{n}\right), \sin\left(\frac{4\pi j}{n}\right), \dots, \sin\left(\frac{2(n-1)\pi j}{n}\right)\right).$$

- Let $P \le \sqrt{\frac{3}{2\pi^2}} \approx 0.389$ & $U_n = \{0, 1, \dots, \lfloor Pn \rfloor\}$ so $\frac{|U_n|}{n} \to P$.
- $\mu_{s_1}(U_n) < P$.
- For p prime all real orthonormal bases are rotations of pairs $\{\vec{c}_j, \vec{s}_j\}$.

Summary

- Families of circulant graphs with prime order have DQUE.
- DQUE does not hold for circulant graphs if we also require real eigenfunctions.
- Can the DQUE condition be improved?

J.M. Harrison and C. Pruss, “A discrete analog of quantum unique ergodicity for circulant graphs,” (In preparation)

M. Magee, J. Thomas and Y. Zhao, “Quantum Unique Ergodicity for Cayley graphs of quasirandom groups,” *CMP* (2023) **402** 3021–3044