The variance of coefficients of the characteristic polynomial of regular quantum graphs

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Dynamical approach to spectral statistics

'71 Gutzwiller's trace formula for the density of states in the semiclassical limit.

'85 Berry - Diagonal approximation to the form factor using Hannay-Ozorio de Almeida sum rule.

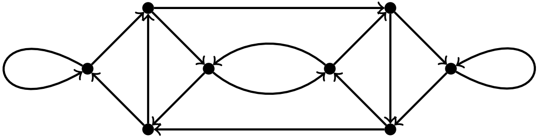
'99 Kottos and Smilansky - trace formula for the density of states of quantum graphs.

'01 Sieber and Richter - 2nd order contribution to the small parameter asymptotics of the form factor from figure 8 orbits with one self-intersection.

'03 Berkolaiko, Schanz and Whitney - 2nd and 3rd order contributions on quantum graphs.

'04 Müller, Heusler, Braun, Haake and Altland - all higher order contributions.

Graphs



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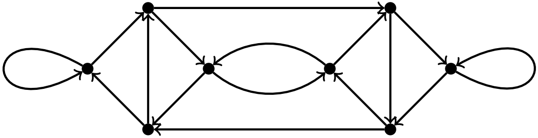

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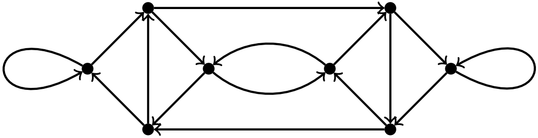

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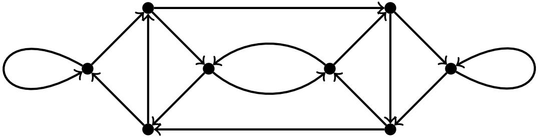

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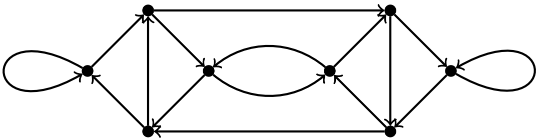

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Quantum evolution op. $\mathbf{U}(k) = \boldsymbol{\Sigma} \mathrm{e}^{\mathrm{i}k\mathbf{L}}$, with $\mathbf{L} = \mathrm{diag}\{L_1, \ldots, L_B\}$, defines a unitary stochastic matrix ensemble - Tanner '01.

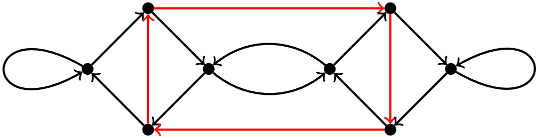
Characteristic polynomial

Characteristic polynomial of $\mathbf{U}(k)$

$$\det\left(\mathbf{U}\left(k\right)-\zeta\mathbf{I}\right)=\sum_{n=0}^{B}a_n(k)\zeta^{B-n}$$

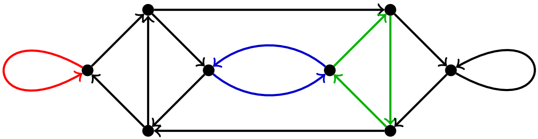
- Secular equation $\det\left(\mathbf{U}\left(k\right)-\mathbf{I}\right)=0$.
- Riemann-Siegel lookalike formula, $a_n = a^*_{B-n}$ – Kottos and Smilansky '99
- Variance of coeffs of characteristic polynomial of binary graphs in semiclassical limit using a diagonal approximation – Tanner '02, Band-Harrison-Sepanski '19

Periodic orbits



- A *periodic orbit* $\gamma = (b_1, \ldots, b_m)$ is the equivalence class of closed paths under cyclic shifts, $t(b_j) = o(b_{j+1})$.
- A *primitive periodic orbit* is a periodic orbit that is not a repetition of a shorter orbit.
- *Topological length* of γ is m.
- *Metric length* of γ is $L_\gamma = \sum_{b_j \in \gamma} L_{b_j}$.
- *Stability amplitude* is $A_\gamma = \Sigma_{b_2 b_1} \Sigma_{b_3 b_2} \ldots \Sigma_{b_m b_{m-1}} \Sigma_{b_1 b_m}$.

Pseudo orbits



- A *pseudo orbit* $\bar{\gamma} = \{\gamma_1, \ldots, \gamma_M\}$ is a set of periodic orbits.
- $m_{\bar{\gamma}} = M$ no. of periodic orbits in $\bar{\gamma}$.
- *Metric length* $L_{\bar{\gamma}} = \sum_{j=1}^{M} L_{\gamma_j}$.
- *Stability amplitude* $A_{\bar{\gamma}} = \prod_{j=1}^{M} A_{\gamma_j}$.
- A *primitive pseudo orbit (PPO)* is a set of distinct primitive periodic orbits.
- $\mathcal{P}^n$ set of PPO with n bonds.

Theorem 1 (Band-Harrison-Joyner '12)

Coefficients of the characteristic polynomial are given by,

$$a_n = \sum_{\bar{\gamma} \in \mathcal{P}^n} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}} e^{ikL_{\bar{\gamma}}} \ .$$

Idea

- Expand $\det(\mathbf{U}(k) - \zeta \mathbf{I})$ as a sum over permutations.
- A permutation $\rho \in S_B$ can contribute iff $\rho(b)$ is adjacent to b for all b in ρ.
- Representing ρ as a product of disjoint cycles each cycle is a primitive periodic orbit.

Variance of coefficients of the characteristic polynomial

$$\langle a_n \rangle = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \langle |a_n|^2 \rangle_k &= \sum_{\bar{\gamma},\bar{\gamma}' \in \mathcal{P}^n} (-1)^{m_{\bar{\gamma}}+m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \lim_{K\to\infty} \frac{1}{K} \int_0^K \mathrm{e}^{\mathrm{i}k(L_{\bar{\gamma}}-L_{\bar{\gamma}'})} \mathrm{d}k \\ &= \sum_{\bar{\gamma},\bar{\gamma}' \in \mathcal{P}^n} (-1)^{m_{\bar{\gamma}}+m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \delta_{L_{\bar{\gamma}},L_{\bar{\gamma}'}} \qquad (3) \end{aligned}$$

Diagonal contribution

$$\langle |a_n|^2 \rangle_{\text{diag}} = \sum_{\bar{\gamma} \in \mathcal{P}^n} |A_{\bar{\gamma}}|^2 = 2^{-n} |\mathcal{P}^n| \ .$$

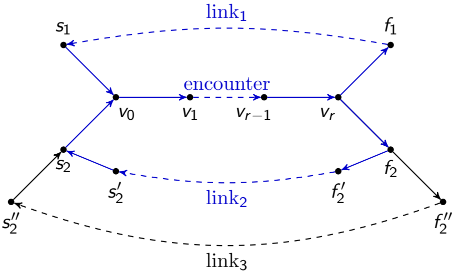
Theorem 2 (Harrison-Hudgins '22)

For a 4-regular quantum graph with $\{L_b\}$ incommensurate,

$$\langle |a_n|^2 \rangle = \frac{1}{2^n} \left(|\mathcal{P}_0^n| + \sum_{N=1}^{n} 2^N \, |\widehat{\mathcal{P}}_N^n| \right) ,$$

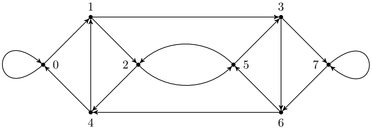
where $\mathcal{P}_0^n \subset \mathcal{P}^n$ with no self-intersections and $\widehat{\mathcal{P}}_N^n \subset \mathcal{P}^n$ with N self-intersections, all of which are 2-encounters of length zero.

Self-intersections



- **2-encounter:** $\bar{\gamma} = (\gamma_1, \ldots, \gamma_m)$ with no self-intersections in $\gamma_2, \ldots, \gamma_m$ and $\gamma_1 = (1, 2)$, link 1 followed by link 2.
- **3-encounter:** Define $\bar{\gamma}$ similarly but with $\gamma_1 = (1, 2, 3)$.
- Encounter *length zero* if it contains no bonds, $v_0 = v_r$.

Example: Binary de Bruijn graph with $B = 2^4$



n	$\lvert\mathcal{P}_0^n\rvert$	$\lvert\widehat{\mathcal{P}}_1^n\rvert$	$\lvert\widehat{\mathcal{P}}_2^n\rvert$	$\langle\lvert a_n\rvert^2\rangle$	Numerics	Error
0	1	0	0	1	1.000000	0.000000
1	2	0	0	1	0.999991	0.000009
2	2	0	0	1/2	0.499999	0.000001
3	4	0	0	1/2	0.499999	0.000001
4	8	0	0	1/2	0.499999	0.000001
5	8	8	0	3/4	0.749998	0.000002
6	8	20	0	3/4	0.749986	0.000014
7	16	16	8	5/8	0.624989	0.000011
8	16	16	24	9/16	0.562501	-0.000001

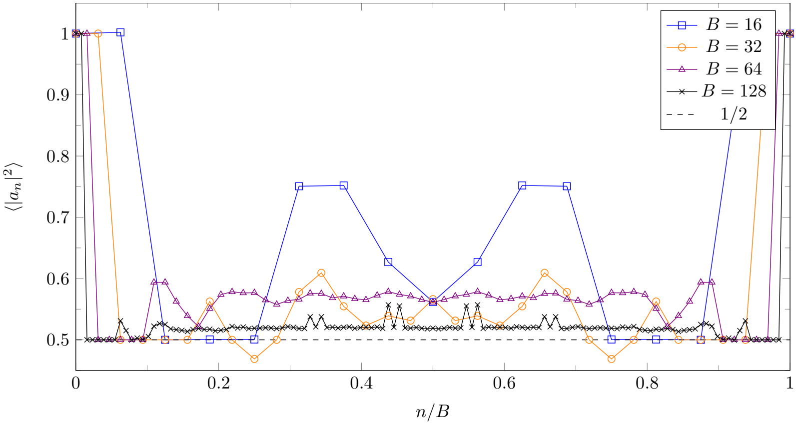
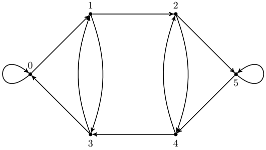


Figure 1: Variance of coefficients of the characteristic polynomial for the family of 4-regular binary de Bruijn graphs with 2^r vertices.

Example: Binary graph with $B = 3 \cdot 2^2$



n	$\lvert\mathcal{P}_0^n\rvert$	$\lvert\widehat{\mathcal{P}}_1^n\rvert$	$\langle\lvert a_n\rvert^2\rangle$	Numerics	Error
0	1	0	1	1.000000	0.000000
1	2	0	1	1.000000	0.000000
2	3	0	3/4	0.750001	-0.000001
3	6	0	3/4	0.750003	-0.000003
4	10	4	7/8	0.874999	0.000001
5	8	4	1/2	0.499998	0.000002
6	8	8	3/8	0.374999	0.000001

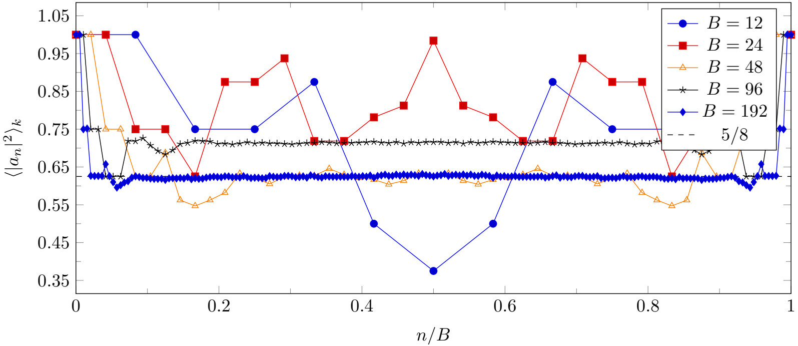


Figure 2: Variance of coefficients of the characteristic polynomial for the family of 4-regular binary graphs with $3 \cdot 2^r$ vertices.

Sketch of a proof of Theorem 2

The sum over PPO can be replaced by a sum over *irreducible pseudo orbits* where no bonds are repeated $\widehat{\mathcal{P}}^n$.

$$\langle |a_n|^2 \rangle = \sum_{\bar{\gamma},\bar{\gamma}' \in \widehat{\mathcal{P}}^n} (-1)^{m_{\bar{\gamma}} + m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \, \delta_{L_{\bar{\gamma}}, L_{\bar{\gamma}'}} = \sum_{\bar{\gamma} \in \widehat{\mathcal{P}}^n} C_{\bar{\gamma}} \tag{4}$$

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- A PPO with an encounter of positive length is not irreducible.
- A PPO with an l-encounter with $l \geq 3$ is not irreducible as there are repeated bonds before and after the encounter.
- A PPO with a single 2-encounter length zero if $\bar{\gamma}' \neq \bar{\gamma}$ then $m_{\bar{\gamma}'} = m_{\bar{\gamma}} \pm 1$ and $\bar{A}_{\bar{\gamma}'} = -A_{\bar{\gamma}}$, hence $C_{\bar{\gamma}} = 2 \cdot 2^{-n}$.

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- To extend results to $2k$-regular graphs requires averaging over assignments of the vertex scattering matrices.

Bibliography

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