Anyons on networks

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Outline

Quantum statistics

Single particle space X.

Two particle statistics - alternative approaches:

- Quantize $X^{\times 2}$ and restrict Hilbert space to the symmetric or anti-symmetric subspace.

$$\psi(x_1, x_2) = \pm\psi(x_2, x_1)$$

Bose-Einstein/Fermi-Dirac statistics.

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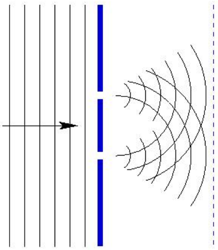
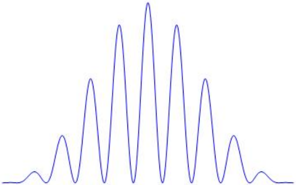
  $$\psi(x_1, x_2) = \pm\psi(x_2, x_1)$$

  Bose-Einstein/Fermi-Dirac statistics.
- (Leinaas and Myrheim '77)
  Treat particles as indistinguishable, $\psi(x_1, x_2) \equiv \psi(x_2, x_1)$.
  Quantize two particle configuration space.

Aharonov-Bohm effect





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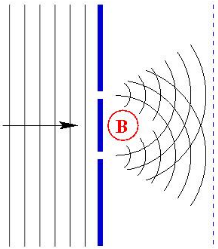
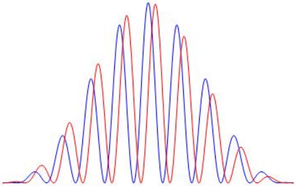
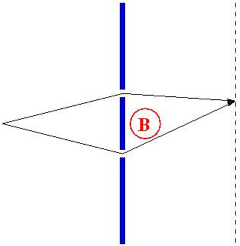


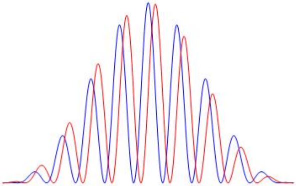


Turn on magnetic field **B** in region inaccessible to particle.

Aharonov-Bohm effect

Path integral formulation.





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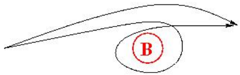
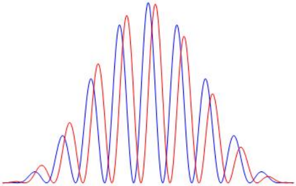
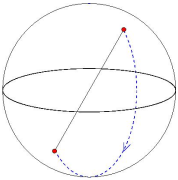




$\mathbf{B} = \nabla \times \mathbf{A}$.
Contribution from paths enclosing $\mathbf{B}$ acquires a phase $e^{i\theta}$ where $\theta = \oint \mathbf{A} . ds$, as $\mathbf{A}$ cannot be zero everywhere on path enclosing $\mathbf{B}$.

Bose-Einstein and Fermi-Dirac statistics

Two indistinguishable particles in $\mathbb{R}^3$. At constant separation relative coordinate lies on projective plane.



Exchanging particles corresponds to rotating relative coordinate around closed loop p.

p is not contractible but p^2 is contractible.

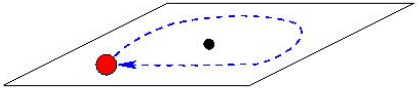
A phase factor $e^{i\theta}$ associated to p requires $(e^{i\theta})^2 = 1$.

Quantizing configuration space with $\theta = \pi$ corresponds to *Fermi-Dirac statistics* and $\theta = 0$ to *Bose-Einstein statistics*.

Anyon statistics

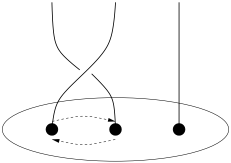
Pair of indistinguishable particles in $\mathbb{R}^2$.

- Particles not coincident.
- Relative position coordinate in $\mathbb{R}^2 \setminus \mathbf{0}$.
- Exchange paths are closed loops about $\mathbf{0}$ in relative coordinate.
- As in the Aharonov-Bohm effect **any** phase factor $e^{i\theta}$ can be associated with a primitive path enclosing $\mathbf{0}$.

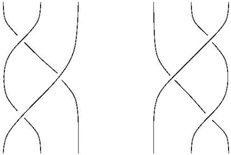


Braid group

For n indistinguishable particles on $\mathbb{R}^2$, σ_j exchanges adjacent particles $j = 1, \ldots, n-1$.



Relations $\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}$ for $j = 1, \ldots, n-2$.



Generates *B_n braid group on n strands*.

A potted history of anyons

(77) Leinaas and Myrheim - quantum mechanics on configuration spaces.

(82) Wilczek - anyons on surfaces.

(82) Tsui and Strömer - fractional quantum Hall effect.

(83) Laughlin wavefunction.

(05) Sarma, Freedman and Nayek - topologically protected qbits.

(08) Kitaev - network models of topological quantum computation.

Definition

Configuration space of n indistinguishable particles in X,

$$C_n(X) = (X^{\times n} - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n | x_i = x_j \text{ for some } i \neq j\}$.

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1st homology groups of $C_n(\mathbb{R}^d)$:

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  2 abelian irreps. corresponding to Bose-Einstein & Fermi-Dirac statistics.

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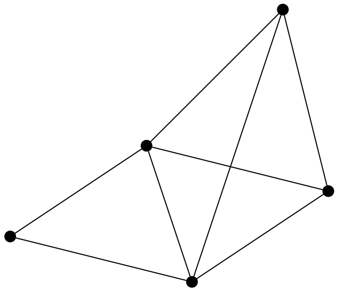
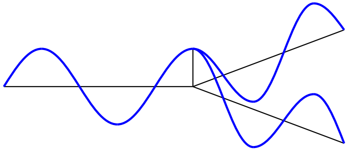
  Abelian irreps. generated by $e^{i\theta}$ – anyon statistics.
- $H_1(C_n(\mathbb{R})) = 1$
  particles cannot be exchanged.

What happens on a network where the underlying space has arbitrarily complex topology?



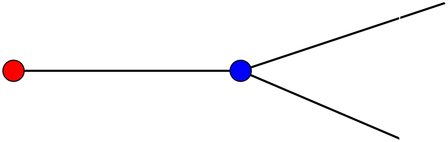
Quantum graphs



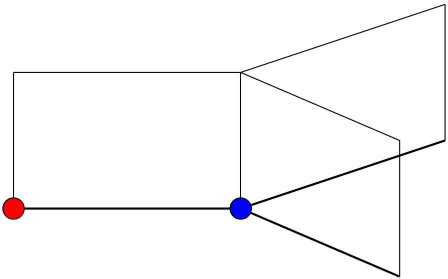
Quantum graphs model phenomena associated with complex quantum systems.

- Free electrons in organic molecules
- Superconducting networks
- Photonic crystals
- Nanotechnology
- Quantum chaos
- Anderson localization

Exchanging indistinguishable particles on a Y



Exchanging indistinguishable particles on a Y



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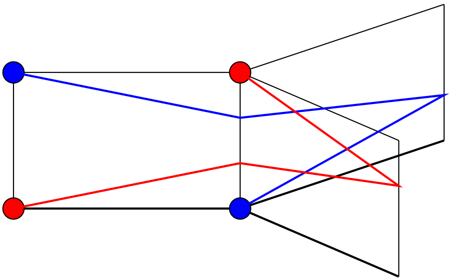
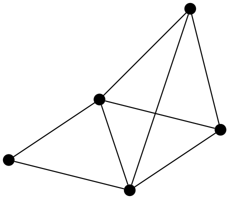


Graph connectivity

- Given a connected graph Γ a *k-cut* is a set of k vertices whose removal makes Γ disconnected.
- Γ is *k-connected* if the minimal cut is size k.
- **Theorem** (Menger) For a k-connected graph there exist at least k independent paths between every pair of vertices.

Example:



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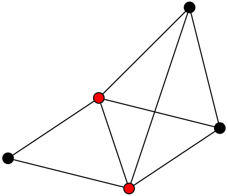

Two cut

Graph connectivity

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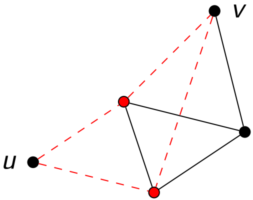

Two independent paths joining u and v.

Features of anyon statistics on networks

3-connected graphs: statistics only depend on whether the graph is planar (Anyons) or non-planar (Bosons/Fermions).

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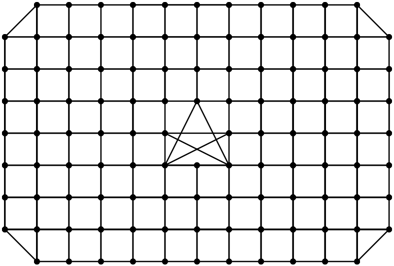

A planar lattice with a small section that is non-planar is locally planar but has Bose/Fermi statistics.

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2-connected graphs: statistics complex but independent of the number of particles.

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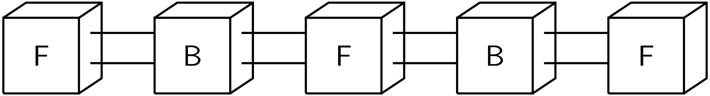

For example, one could construct a chain of 3-connected non-planar components where particles behave with alternating Bose/Fermi statistics.

Features of anyon statistics on networks

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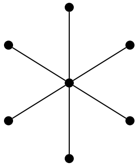
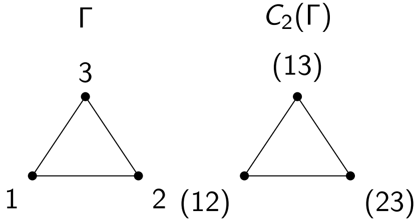
Example, star with E edges.



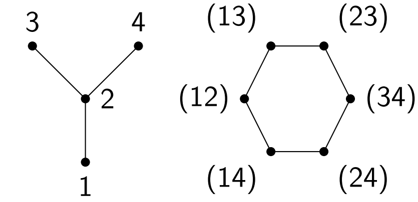
no. of anyon phases

$$\binom{n+E-2}{E-1}(E-2)-\binom{n+E-2}{E-2}+1\ .$$

Basic cases



Exchange of 2 particles around loop c; one free phase ϕ_{c2}.



Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

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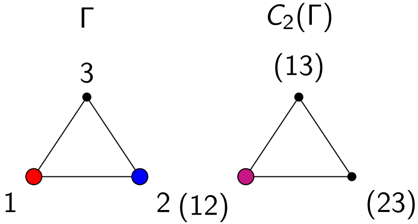

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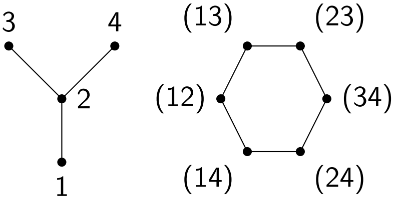

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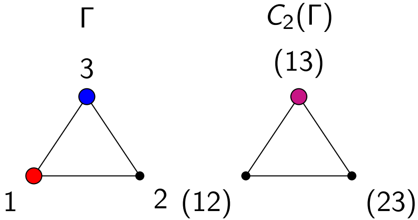

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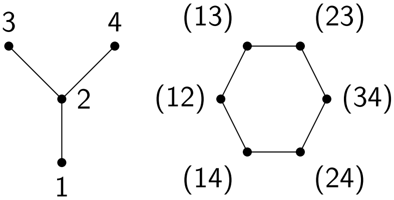

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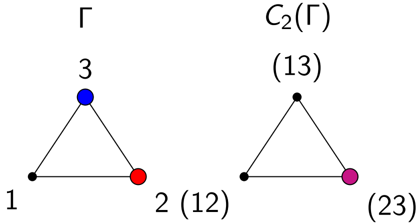

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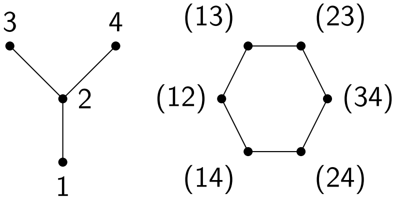

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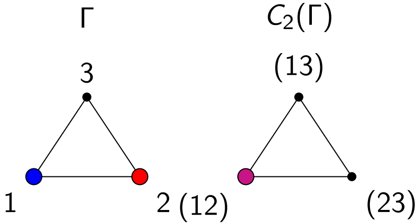

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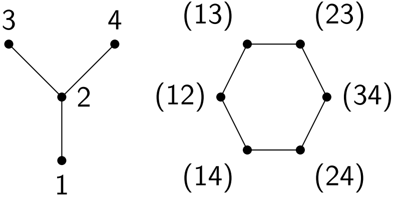

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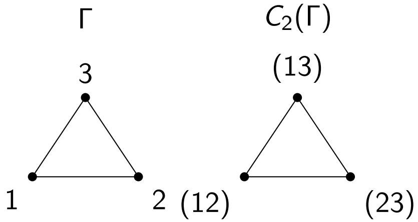

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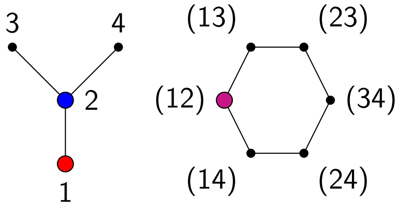

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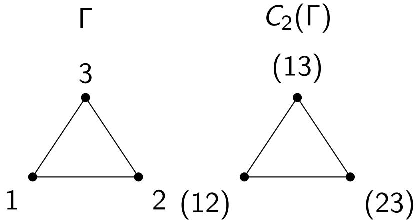

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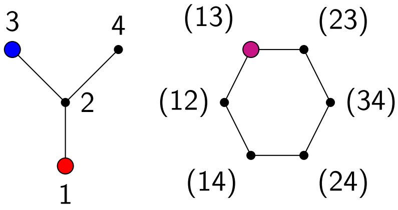

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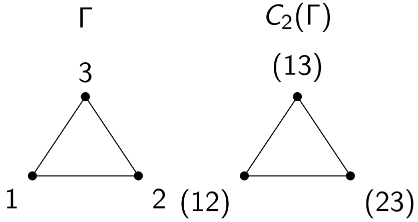

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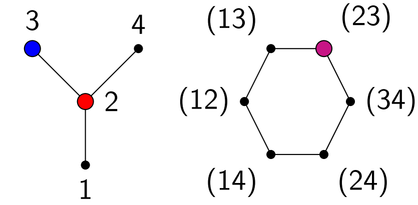

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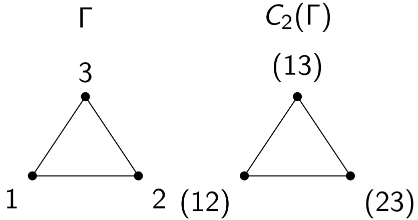

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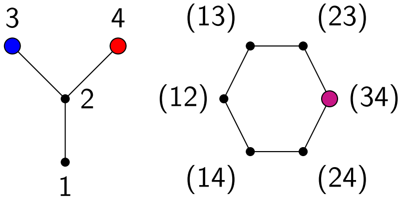

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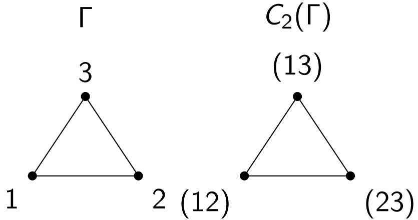

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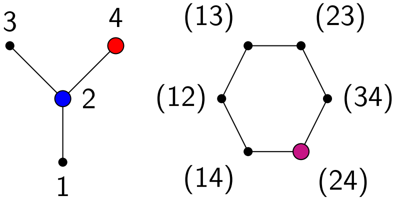

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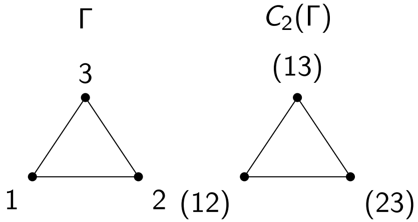

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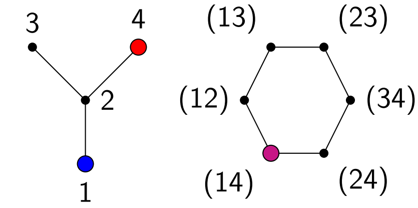

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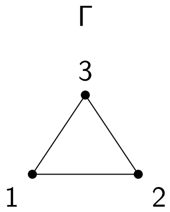

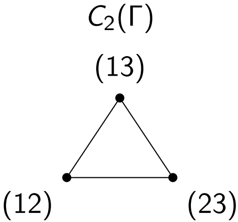

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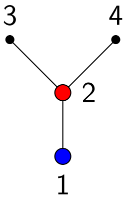

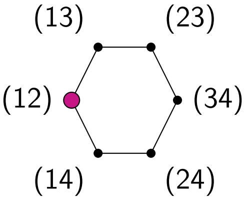

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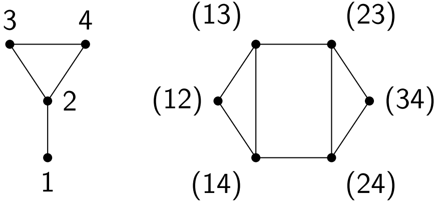
Lasso graph



Identify three 2-particle cycles:

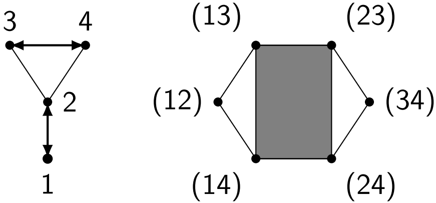
(i) Rotate both particles around loop c; phase $\phi_{c,2}$.

(ii) Exchange particles on Y-subgraph; phase ϕ_Y.

(iii) Rotate one particle around loop c other particle at vertex 1; $(1,2) \to (1,3) \to (1,4) \to (1,2)$, phase $\phi^1_{c,1}$.

Relation from contactable 2-cell $\phi_{c,2} = \phi^1_{c,1} + \phi_Y$.

Lasso graph



Identify three 2-particle cycles:

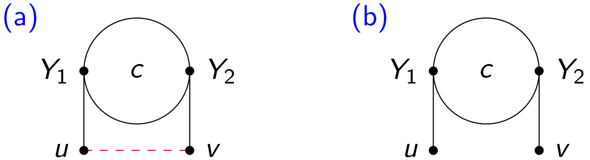
(i) Rotate both particles around loop c; phase $\phi_{c,2}$.

(ii) Exchange particles on Y-subgraph; phase ϕ_Y.

(iii) Rotate one particle around loop c other particle at vertex 1; $(1,2) \to (1,3) \to (1,4) \to (1,2)$, phase $\phi^1_{c,1}$.

Relation from contactable 2-cell $\phi_{c,2} = \phi^1_{c,1} + \phi_Y$.

Let c be a loop. What is the relation between $\phi^u_{c,1}$ and $\phi^v_{c,1}$?



(a) u and v joined by path disjoint from c.
$\phi^u_{c,1} = \phi^v_{c,1}$ as exchange cycles homotopy equivalent.

(b) u and v *only* joined by paths through c.
Two lasso graphs so $\phi_{c,2} = \phi^u_{c,1} + \phi_{Y_1}$ & $\phi_{c,2} = \phi^v_{c,1} + \phi_{Y_2}$.
Hence $\phi^u_{c,1} - \phi^v_{c,1} = \phi_{Y_2} - \phi_{Y_1}$.

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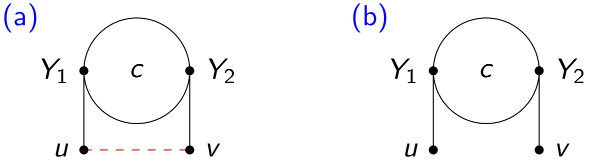

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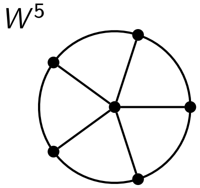
Hence $\phi^u_{c,1} - \phi^v_{c,1} = \phi_{Y_2} - \phi_{Y_1}$.

- Relations between phases involving c encoded in phases ϕ_Y. $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where A determined by Y-cycles.
- In (a) we have a $\mathcal{B}$ subgraph & using (b) also $\phi_{Y_1} = \phi_{Y_2}$.

3-connected graphs

The prototypical 3-connected graph is a *wheel* W^k.



Theorem (Wheel theorem)

Let Γ be a simple 3-connected graph different from a wheel. Then for some edge $e \in \Gamma$ either $\Gamma \setminus e$ or Γ/e is simple and 3-connected.

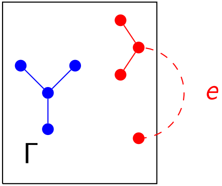
- $\Gamma \setminus e$ is Γ with the edge e removed.
- Γ/e is Γ with e contracted to identify its vertices.

Lemma

For 3-connected simple graphs all phases ϕ_Y are equal up to a sign.

Sketch proof. The lemma holds on K_4 (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases ϕ_Y are the same as an original phase.

Adding an edge: $\Gamma \cup e$



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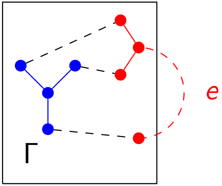

Using 3-connectedness identify independent paths in Γ to make $\mathcal{B}$. Then $\phi_Y = \phi_Y$.

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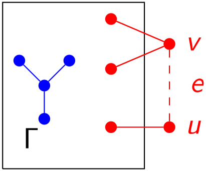
Vertex expansion: Split vertex of degree > 3 into two vertices u and v joined by a new edge e.



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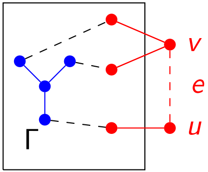

Using 3-connectedness identify independent paths in Γ to make $\mathcal{B}$. Then $\phi_{\color{red}Y} = \phi_{\color{blue}Y}$.

Theorem

For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$*, where* $A = \mathbb{Z}_2$ *for non-planar graphs and* $A = \mathbb{Z}$ *for planar graphs.*

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For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where $A = \mathbb{Z}_2$ for non-planar graphs and $A = \mathbb{Z}$ for planar graphs.

Proof.

- For K_5 and $K_{3,3}$ every phase $\phi_Y = 0$ or π. By Kuratowski's theorem a non-planar graph contains a subgraph which is isomorphic to K_5 or $K_{3,3}$.

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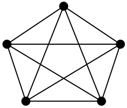
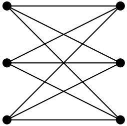
- For planar graphs the anyon phase can be introduced by drawing the graph in the plane and integrating the anyon vector potential $\frac{\alpha}{2\pi}\hat{z} \times \frac{r_1 - r_2}{|r_1 - r_2|^2}$ along the edges of the two-particle graph.

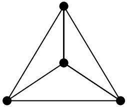
Examples



K_5: 6 A-B phases, 1 discrete phase of 0 or π.



$K_{3,3}$: 4 A-B phases, 1 discrete phase of 0 or π.



K_4: 3 A-B phases, 1 anyon phase.

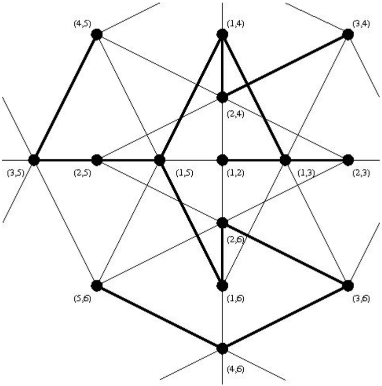


Figure: Configuration space graph $C_2(K_{3,3})$, edges shown as solid lines are in a spanning subtree with root $(1, 2)$. Open edges are joined left to right and top to bottom.

Classification of graph statistics

Ko & Park (2011)

$$H_1(C_n(\Gamma)) = \mathbb{Z}^{N_1(\Gamma)+N_2(\Gamma)+N_3(\Gamma)+\beta_1(\Gamma)} \oplus \mathbb{Z}_2^{N_3'(\Gamma)}$$

- $N_1(\Gamma)$ sum over one cuts j of $N(n,\Gamma,j)$.

$$N(n,\Gamma,j) = \binom{n+\mu_j-2}{n-1}(\mu(j)-2) - \binom{n+\mu_j-2}{n} - (v_j - \mu_j - 1)$$

  μ_j # components of $\Gamma \setminus j$.

- $N_2(\Gamma)$ sum over two connected components of Γ.
- $N_3(\Gamma)$ # 3-connected planar components of Γ.
- $N_3'(\Gamma)$ # 3-connected non-planar components of Γ.
- $\beta_1(\Gamma)$ # of loops of Γ.

Summary

- Classification of abelian quantum statistics on graphs via graph theoretic argument.
- Physical insight into dependence of statistics on graph connectivity.
- Identified new features of anyon statistics.
- Are there phenomena associated with new forms of anyon behavior - e.g. fractional quantum Hall experiment on network?

JH, JP Keating, JM Robbins and A Sawicki, "*n*-particle quantum statistics on graphs," *Commun. Math. Phys.* (2014) **330** 1293–1326 `arXiv:1304.5781`

JH, JP Keating and JM Robbins, "Quantum statistics on graphs," *Proc. R. Soc. A* (2010) doi:10.1098/rspa.2010.0254 `arXiv:1101.1535`