Quantum graphs with symmetry

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Outline

Circulant graphs

- Cayley graph of cyclic group $\mathbb{Z}_n$.
- Vertices $\{1, \ldots, n\}$.
- Fix $\vec{a} = (a_1, \ldots, a_d)$, st $0 < a_1 < a_2 < \cdots < a_d < n/2$.
- Edges $(i,j) \in \mathcal{E}$ iff $|i-j| \equiv a_h \mod n$; circulant graph $C_n(\vec{a})$.
- $C_n(\vec{a})$ connected iff $\gcd(a_1, \ldots, a_d, n) = 1$.

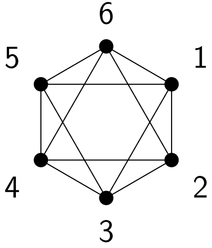

Figure: Circulant graph $C_6(1,2)$.

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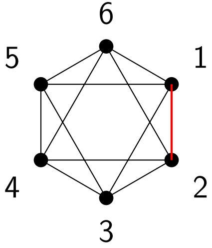

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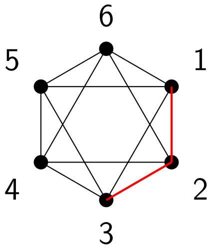

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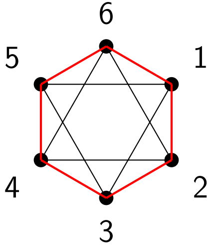

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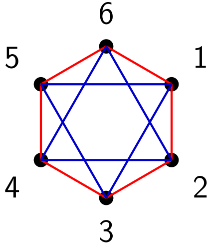


Figure: Circulant graph $C_6(1, 2)$.

- Circulant graphs share rotation symmetry with star graphs.
- Let $\sigma(j) = j + 1$; $i \sim j$ iff $\sigma(i) \sim \sigma(j)$.

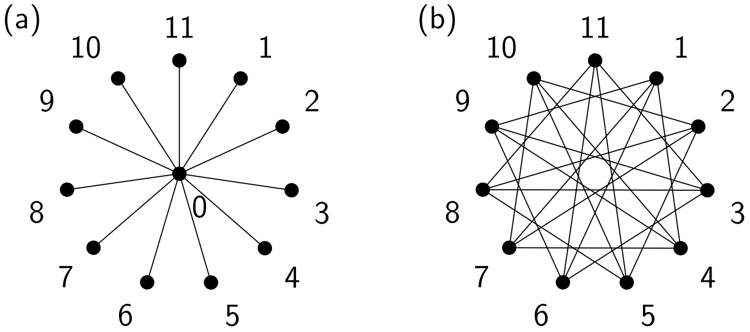


Figure: (a) Star graph with 11 edges. (b) Circulant graph $C_{11}(3, 5)$.

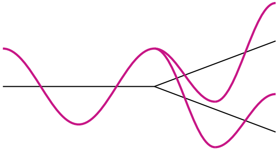
Quantum graph

- *Metric graph:* associate edge (i,j) with interval $[0, L_{i,j}]$.
- *Total length:* $\mathcal{L} = \sum_{(i,j)\in\mathcal{E}} L_{i,j}$.
- *Laplace equation* on $[0, L_{i,j}]$,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x_{i,j}^2}\psi_{i,j}(x_{i,j}) = k^2\psi_{i,j}(x_{i,j}) \ .$$

- *Domain* $H^2[0, L_{i,j}]$ on each edge with Neumann-like vertex conditions; ψ continuous and

$$\sum_{j\sim i}\psi'_{i,j}(i) = 0 \ . \qquad (1)$$



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Quantum circulant graphs

1. $C_n(\vec{L};\vec{a})$ *quantum circulant graph* edge lengths $\vec{L}$.
2. $C_n(\vec{l};\vec{a})$ *quantum circulant graph symmetric edge lengths;* fix $\vec{l} = \{l_1, \dots, l_d\}$, assign edge lengths to $C_n(\vec{L};\vec{a})$ s.t. (i,j) has length l_h when $|i-j| \equiv a_h \mod n$.

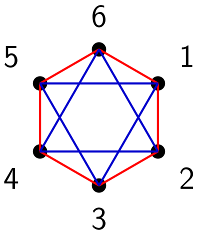

Figure: Quantum circulant graph symmetric edge lengths $C_6(\vec{l};1,2)$.

Secular equation

- $\mathbf{\Phi} = (\phi_1, \ldots, \phi_n)^T$ values of ψ at vertices,

$$\psi_{i,j}(x) = \left(\frac{\phi_j - \phi_i \cos kL_{i,j}}{\sin kL_{i,j}} \right) \sin kx + \phi_i \cos kx \ .$$

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- Substitute in vertex condition (1) at i,

$$\sum_{j \sim i} (\phi_j \csc kL_{i,j} - \phi_i \cot kL_{i,j}) = 0 \ .$$

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- In matrix form $M(k)\mathbf{\Phi} = \mathbf{0}$,

$$[M]_{ii} = -\sum_{j \sim i} \cot kL_{i,j}$$

$$[M]_{ij} = \csc kL_{i,j} \qquad i \sim j$$

Theorem 1 (Secular equation)

Let $k \in \mathbb{C} \setminus \mathcal{D}$. Then k^2 is an eigenvalue of the negative Laplace op. on $C_n(\vec{L}; \vec{a})$ with multiplicity m iff k is an m'th root of,

$$\det M(k) = 0 \ .$$

Dirichlet spectrum

$$\mathcal{D} = \left\{ \frac{m\pi}{L_{i,j}} : m \in \mathbb{N}, (i,j) \in \mathcal{E} \right\}$$

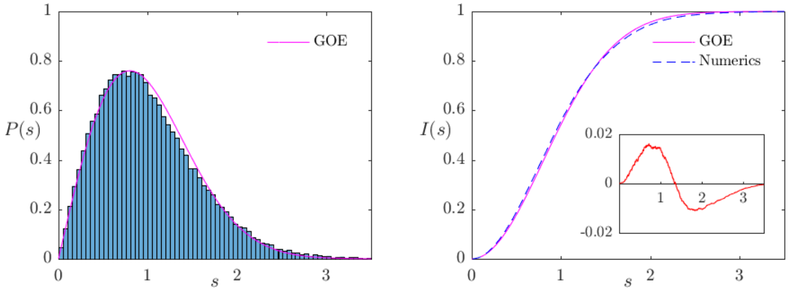

Figure: Histogram of nearest-neighbor spacing distribution, 50, 036 eigenvalues of $C_{49}(\vec{L}; (3, 4, 9, 12, 15, 19, 20))$ compared to Wigner surmise for the GOE and corresponding integrated nearest-neighbor spacing distribution.

Circulant matrix

$$C = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}$$

Adjacency matrix of circulant graph is circulant matrix.

Representer

$$p(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}$$

$$\det C = \prod_{j=0}^{n-1} p\left(\omega^j\right) \qquad \omega = \exp\left(2\pi \mathrm{i}/n\right)$$

Secular equation with symmetric edge lengths

Consider secular equation $\det M(k) = 0$ for $C_n(\vec{l};\vec{a})$ with symmetric edge lengths.

$$[M]_{ii} = -\sum_{h=1}^{d} \cot kl_h$$
$$[M]_{ij} = \csc kL_h \qquad |i-j| = a_h \mod n$$

Let A_h be the adjacency matrix of the subgraph $C_n(a_h)$.

$$M(k) = \left(-2\sum_{h=1}^{d} \cot k\ell_h\right) \mathrm{I}_n + \sum_{h=1}^{d} (\csc k\ell_h)\, A_h$$

So $M(k)$ is a circulant matrix.

Theorem 2 (Secular equation with symmetric edge lengths)

Let n odd and $k \in \mathbb{C} \setminus \mathcal{D}$. Then $\lambda = k^2 > 0$ is an eigenvalue of negative Laplace op. on $C_n(\vec{l}; \vec{a})$ with multiplicity m iff k is an m'th root of,

$$p_0(k) \prod_{j=1}^{(n-1)/2} |p_j(k)|^2 = 0 \ .$$

$$p_0(k) = \sum_{h=1}^{d} \tan\left(\frac{kl_h}{2}\right)$$

$$p_j(k) = \sum_{h=1}^{d} \cos\left(\frac{2\pi j a_h}{n}\right) \csc(kl_h) - \cot(kl_h) \qquad j \geq 1$$

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$$p_0(k) = \sum_{h=1}^{d} \tan\left(\frac{kl_h}{2}\right) = 0 \qquad \text{star graph}$$

$$p_j(k) = \sum_{h=1}^{d} \cos\left(\frac{2\pi j a_h}{n}\right) \csc(kl_h) - \cot(kl_h) \qquad j \geq 1$$

Quotient graph

Isolate subspace associated to irreducible representation of graph symmetry.

- Isospectral graphs - Band, Parzanchevski, Ben-Shach (‘09), Parzanchevski, Band (‘10).
- GSE statistics without spin - Joyner, Müller, Sieber (‘14).

Quotient graph of circulant graph

- Add dummy vertices at $l_h/2$.
- *Fundamental domain*; vertex & neighbouring dummy vertices.
- *Quotient graph*; identify pairs of dummy vertices of fundamental domain on edges length $l_h/2$.

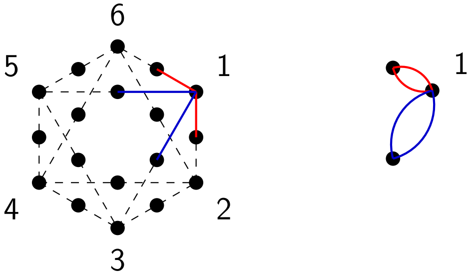


Figure: $C_6(\vec{l}; 1, 2)$ with dummy vertices and quotient graph.

- Symmetry group $\mathbb{Z}_n$ generated by $\sigma(v) = v + 1$.
- Irreps. $\mathcal{S}_j$ for $j = 0, \ldots, n-1$; $\mathcal{S}_j(\sigma) = \mathrm{e}^{\mathrm{i}\theta_j}$ where $\theta_j = 2\pi j/n$.
- Spectrum decomposes into subspectra whose eigenfunctions transform according to $\mathcal{S}_j$.

Subspectrum transforming according to $\mathcal{S}_j$ is spectrum of quotient graph with matching conditions at h'th degree two vertex,

$$\psi_{h-}(l_h/2) = \mathrm{e}^{\mathrm{i}a_h\theta_j}\psi_{h+}(l_h/2) \ ,$$
$$\psi'_{h-}(l_h/2) = \mathrm{e}^{\mathrm{i}a_h\theta_j}\psi'_{h+}(l_h/2) \ .$$

ψ_{h-} function on $[0, l_h/2]$ and ψ_{h+} function on $[l_h/2, \ell_h]$.

Let ϕ be value of ψ at central vertex of quotient graph,

$$\psi_{h-}(x) = \phi \left[\cos(kx) + (\mathrm{e}^{\mathrm{i}a_h\theta_j}\csc(kl_h) - \cot(kl_h))\sin(kx)\right]$$
$$\psi_{h+}(x) = \phi \mathrm{e}^{-\mathrm{i}a_h\theta_j}\left[\cos(kx) + (\mathrm{e}^{\mathrm{i}a_h\theta_j}\csc(kl_h) - \cot(kl_h))\sin(kx)\right]$$

Neumann-like condition at central vertex requires,

$$\sum_{h=1}^{d} \cos(a_h\theta_j)\csc(k\ell_h) - \cot(k\ell_h) = 0 \ . \qquad (2)$$

Equivalently $p_j(k) = 0$.

(Note: $p_j(k) = p_{n-j}(k)$ as $\theta_{n-j} = 2\pi - \theta_j$.)

Dirichlet spectrum

Eigenvalues in Dirichlet spectrum of $C_n(\vec{l}; \vec{a})$ also appear with Neumann-like vertex conditions.

Lemma 3

For a.e. $\vec{l} \in (1-\epsilon, 1+\epsilon)^d$ and all $m \in \mathbb{N}$, $m^2\pi^2/l_h^2$ is in the spectrum of the graph with multiplicity $|J|$ where,

$$J = \{j \in \{0, \dots, n-1\} : 2ja_h = qn \\ \textit{for some odd/even } q \textit{ when } m \textit{ is odd/even}\} .$$

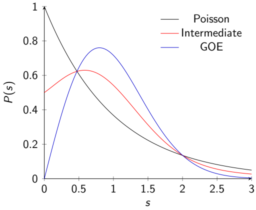
- For any n, $m^2\pi^2/\ell_g^2$ is in the spectrum for even m.
- For odd n, $m^2\pi^2/\ell_g^2$ is not in the spectrum for odd m.

Proof. Use quotient graph with $k = m\pi/l_h$.

Intermediate statistics

Spectral statistics between RMT ensembles and Poisson.

- Šeba billiards - Šeba ('90), Alveverio, Šeba ('91).
- Aharonov-Bohm int. billiards - Date, Jain, Murthy ('94), Bolgomolny, Giraud, Schmidt ('01), Rahav, Fishman ('01).
- Polygonal billiards rational angles - Parab, Jain ('96), Grémaud, Jain ('98), Bogomolny, Gerland, Schmidt ('01).
- Quantum map eigenphases - Giraud, Marklof, O'Keefe ('04).
- Anderson model at metal-insulator transition point.



Two-point correlation function

Two-point correlation function $R_2(x)$ defined by,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} g(k_i - k_j) = g(0) + \int_{-\infty}^{\infty} g(x) R_2(x) \, \mathrm{d}x$$

for suitable test function g.

Note: if g approximates characteristic function of an interval R_2 is a measure of pairs of eigenvalues whose separation falls in the interval.

Star graph

- Bogomolny, Gerland & Schmit ('99,'01)

$$R_2(x) \sim \frac{\pi\sqrt{3}}{2}x \qquad x \to 0$$

- Berkolaiko & Keating ('99), Berkolaiko, Bogomolny & Keating ('01)

$$R_2(x) \sim 1 + \frac{2}{\pi^2 x^2} + \frac{76}{\pi^4 x^4} + O\left(\frac{1}{x^6}\right) \qquad x \to \infty$$

- Dirac op. on rose graph - H & Winn ('12)

$$R_2(x) \sim \frac{\pi c}{6}x \qquad x \to 0$$

$$R_2(x) \sim 1 + \frac{2}{\pi^2 x^2} - \frac{13}{8\pi^4 x^4} + O\left(\frac{1}{x^6}\right) \qquad x \to \infty$$

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Small parameter asymptotic

$$\mathbb{E}(R_2(x)) \sim \frac{1}{\pi} \ln^2 \left(\frac{x}{c} \right) x \qquad x \to 0$$

- Secular eqn. similar to star graph (set $\theta_h = 0$),

$$\sum_{h=1}^{d} \cos(\theta_h) \csc(kl_h) - \cot(kl_h) = 0$$

$$\cos\theta \csc k - \cot k = \sum_{m=-\infty}^{\infty} ((-1)^m \cos\theta - 1) \left(\frac{1}{k + \pi m} - \frac{m\pi}{1 + m^2\pi^2} \right)$$

- Statistics of small spacing's approximated by zeros of,

$$\frac{r_1}{k - c_1} + \frac{r_2}{k - c_2} + \frac{r_3}{k - c_3} = 0 \ .$$

- $r_i = 1 - \cos\theta_i$; θ_i randomly chosen a_h multiples of $2\pi j/n$.

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Large parameter asymptotic

$$\mathbb{E}(R_2(x)) \sim 1 + \frac{2}{\pi^2 x^2} - \frac{1}{2\pi^4 x^4} + O\left(\frac{1}{x^6}\right) \quad x \to \infty$$

- Use small parameter behavior of *form factor*: Fourier transform of $R_2(x)$,

$$\mathbb{E}\left(K(\tau)\right) = \lim_{\substack{d\to\infty \\ t/2d\to\tau}} \sum_{h=1}^{d} \tilde{K}_h(t,d)$$

$$\tilde{K}_h(t,d) = \frac{d}{2\mathcal{L}^2} \sum_{\substack{L \text{ restricted} \\ \text{to } h \text{ edges}}} L^2 \mathbb{E}\left(\sum_{\substack{p\in\mathcal{P}_t \\ L_p = L}} \frac{A_p}{r_p}\right)^2$$

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Periodic orbits on one edge: $h = 1$

$$\tilde{K}_1(t,d) = \frac{d}{2\mathcal{L}^2} \sum_{\substack{L \text{ restricted} \\ \text{to } 1 \text{ edge}}} L^2 \mathbb{E}\left(\sum_{\substack{p \in \mathcal{P}_t \\ L_p = L}} \frac{A_p}{r_p} \right)^2$$

- Scattering coefficients center; $1/d$ or $1/d - 1$ back scattering.
- Scattering dummy vertex; $\mathrm{e}^{\pm \mathrm{i}\theta_e}$, no back scattering.
- Orbit length t even and maximum back scattering

$$p = e\bar{e}e\bar{e} \cdots e\bar{e}e\bar{e}$$

$$\tilde{K}_1(t,d) \approx 2\left(1 - \frac{1}{d}\right)^{2t}$$

$$\lim_{\substack{d \to \infty \\ t/2d \to \tau}} \tilde{K}_1(t,d) \approx \lim_{d \to \infty} 2\left(1 - \frac{1}{d}\right)^{4d\tau} = 2\mathrm{e}^{-4\tau}$$

Periodic orbits on one edge: $h = 1$

- Orbit length t odd and maximum back scattering

$$p = \begin{cases} e\bar{e}e\bar{e}\cdots e\bar{e}\bar{e} \\ \bar{e}e\bar{e}e\cdots \bar{e}ee \end{cases}$$

$$\tilde{K}_1(t,d) \approx \frac{t^2}{2d^2}\left(1-\frac{1}{d}\right)^{2t-2} \mathbb{E}\left\{(\mathrm{e}^{\pm \mathrm{i}\theta_e} + \mathrm{e}^{\mp \mathrm{i}\theta_e})^2\right\}$$

$$\lim_{\substack{d\to\infty \\ t/2d\to\tau}} \tilde{K}_1(t,d) \approx \lim_{d\to\infty} 2\tau^2 \left(1-\frac{1}{d}\right)^{4d\tau-2} \mathbb{E}\left\{4\cos^2\theta_e\right\}$$

$$= 4\tau^2\mathrm{e}^{-4\tau}$$

- Average contribution from t even and t odd,

$$(1+2\tau^2)\mathrm{e}^{-4\tau}$$

Periodic orbits on $h > 1$ edges

- Maximal back scattering; h transitions between edges.
- Choose $h-1$ transition points from $t-1$ locations.
- Scattering at central vertex,

$$\left(\frac{1}{d}-1\right)^{t-h}\frac{1}{d^j} \ .$$

- Phase from r edges with odd bounces,

$$2^{2(h-r)}\mathbb{E}\left\{2^{2r}\cos^2\theta_{e_{i_1}}\cdots\cos^2\theta_{e_{i_r}}\right\} = 2^{2h-r} \ .$$

- Combine results for all h,

$$\mathbb{E}\left(K(\tau)\right) \approx \left(1-\tau-4\tau^2\right)\mathrm{e}^{-4\tau} + \tau\mathrm{e}^{2\tau} \ .$$

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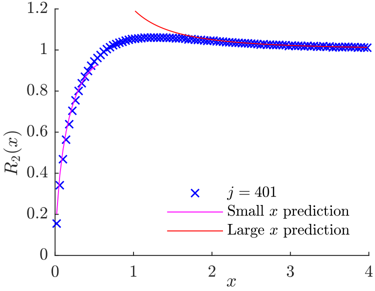


Figure: Two-point correlation function for subspectrum of $C_{7919}(\vec{l}; \vec{a})$ transforming according to irrep. $\mathcal{S}_{401}$, 20 000 089 evals., $c = 5.51$ (left) and two-point correlation function of subspectra of $C_{7919}(\vec{l}; \vec{a})$ averaged over all non-trivial irreps., 5 362 evals. per spectrum, $c = 5.26$ (right).

Cayley graph

- G finite group generated by $\{a, b\}$.
- a, b order ≥ 3, $b \notin \{a, a^{-1}\}$.
- Example: $G = \mathrm{SL}(2, p)$, $p \geq 5$ prime.

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

- Γ Cayley graph of G wrt $S = \{a, a^{-1}, b, b^{-1}\}$,

$$V = G \qquad E = \{(g, gs) : g \in G, s \in S\}$$

- Metric graph: length 2α for edges (g, gs) for $s \in \{a, a^{-1}\}$, length 2β for edges (g, gs) for $s \in \{b, b^{-1}\}$.
- Quantum graph: negative Laplace op. on metric graph with Neumann-like vertex conditions.

Quotient graph

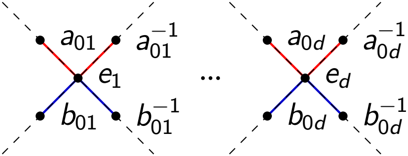
- Given d dim irrep. π,

$$A = \pi(a) \qquad B = \pi(b) \ .$$

- Add dummy vertices a_0, a_0^{-1}, b_0, b_0^{-1} at midpoints.
- Fundamental domain about vertex e; four edges,

$$(e, a_0) \quad (e, a_0^{-1}) \quad (e, b_0) \quad (e, b_0^{-1}) \ .$$

- Take d fundamental domains.
- Identify $a_{01}, \dots a_{0d}, a_{01}^{-1}, \dots a_{0d}^{-1}$, similarly for b vertices.



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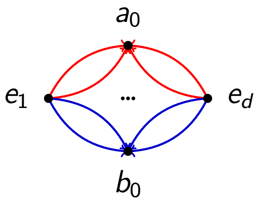

Vertex conditions

- a maps a_0^{-1} to a_0 on Γ.
- h_j function on (e, a_{0j}^{-1}) and f_j function on (e, a_{0j})
- At vertex a_0 Neumann-like conditions require,

$$h_j(\alpha) = \sum_{k=1}^{d} A_{jk} f_k(\alpha) \ ,$$

$$h_j'(\alpha) = -\sum_{k=1}^{d} A_{jk} f_k'(\alpha) \ .$$

- Similarly B defines conditions at b_0.
- Neumann-like conditions at $e_1, \ldots, e_d$.

Example: $\mathrm{SL}(2,7)$

$G = \mathrm{SL}(2,7)$ with 6-dim irrep. generated by,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

Secular equation,

$$-\frac{1}{4}\csc(\alpha k)\csc(2\alpha k)\csc^2(2\beta k)\sin(\beta k) \times \Big(\sin((\alpha+\beta)k) + 4\sin(3(\alpha+\beta)k) + 2\sin((\alpha+3\beta)k) + 4\sin((3\alpha+5\beta)k)\Big) = 0\ .$$

Replace vertices e_j with subgraphs to remove factorization.

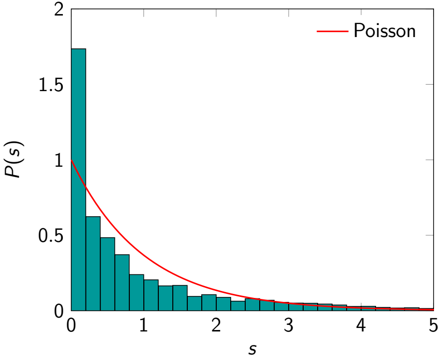

Figure: Histogram of nearest-neighbor spacings for $\mathrm{SL}(2,7)$ example with K_4 subgraph.

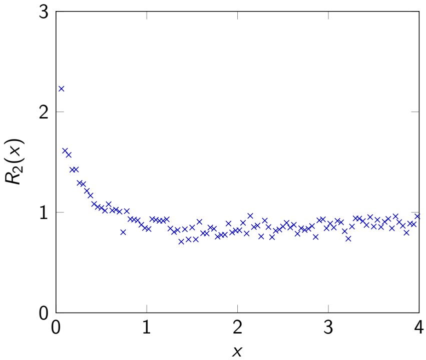

Figure: Two-point correlation function for $\mathrm{SL}(2,7)$ example with K_4 subgraph.

Summary

- Secular equations of circulant graphs exhibit features of star graph equation.
- Intermediate statistics of subspectra transforming according to typical irreps. differs from those associated to trivial representation (and star graph).
- Results for spectral zeta function, spectral determinant and vacuum energy of circulant graphs.

J.M. Harrison and E. Swindle, "Spectral properties of quantum circulant graphs," *J. Phys. A: Math. Theor.* (2019) https://doi.org/10.1088/1751-8121/ab22f3 arXiv:1810.08664

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