Uncertainty, Investor Sentiment, and Innovation*

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Abstract

We develop a theory of innovation waves, investor sentiment, and merger activity based on Knightian uncertainty. Uncertainty-averse investors are more optimistic on an innovation when they can make contemporaneous investments in multiple uncertain projects. Innovation waves occur when there is a critical mass of innovative companies, and are characterized by stronger investor sentiment, higher equity valuation, and hot IPO markets. Our approach to investor sentiment is not based on erroneous beliefs disjoint from economic fundamentals, but depends on uncertainty on the fundamentals. Our model can explain sector-specific booms uncorrelated with aggregate economic activity and the overall stock market.

Keywords: Investor Sentiment, Ambiguity Aversion, Innovation, Hot IPO Markets

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Innovation is arguably one of the most important value drivers in modern corporations and a key source of economic growth (Solow, 1957). There are times when innovation stagnates, but others when technology leaps forward in innovation waves. These spurts of innovation activity are often associated with higher stock market valuations for technology firms, strong (or "optimistic") investor sentiment (Perez, 2002, and Baker and Wurgler, 2007), a strong market for Initial Public Offerings, IPO, (Lee, Shleifer, and Thaler, 1991, and Baker and Wurgler, 2000) and an active market for mergers and acquisitions, M&A (Celikyurt et al., 2010, and Bena and Li, 2014). Interestingly, such flare-ups in innovation and IPO activity are often confined to specific technology sectors, with little or no correlation with the broader equity markets.[1]

While time-varying investor sentiment is often identified as an important cause of the alternating periods of booms and busts, we have a limited understanding of its economic drivers. We propose a novel theory of belief formation, which we identify as "investor sentiment," based on decision theory. We develop our notion of investor sentiment in the context of innovation. Innovation, by its nature, is characterized by a limited knowledge of the relevant probability distributions, a situation best described as "Knightian uncertainty" (Knight, 1921): investors must typically decide whether to fund a project with very limited knowledge of the odds of success.

We show uncertainty aversion can cause investor beliefs to fluctuate (endogenously) between periods of pessimism toward innovative ventures ("cold markets") and periods of greater optimism ("hot markets"). These alternating phases spur innovation waves associated with higher stock market valuations, greater volumes of IPOs, and an active M&A market for technology companies. Our approach can explain the emergence of sector-specific hot and cold markets uncorrelated with aggregate economic factors, such as overall economic activity or stock market performance.

There are many reasons why innovation develops in waves. These include fundamentals such as random scientific breakthroughs with technological spillovers. In this paper, we focus on the interaction between financial markets and the incentives to innovate. We study an economy with multiple entrepreneurs endowed with risky project-ideas that, if successful, may lead to innovations. The innovation process consists of two stages. In the first stage, entrepreneurs must decide whether or not to invest personal resources, such as exerting effort, to innovate. If the first stage is successful,

[1] For example, in the boom years of 1998-2000, the NASDAQ index, which is dominated by technology companies, more than doubled while the general market, as measured for example by the S&P500 index, remained stable.

further development of the innovation requires additional monetary investment. Entrepreneurs raise funds by selling shares of their firms to uncertainty-averse investors. The second stage of the innovation process is uncertain: outside investors are uncertain on the exact distribution of the residual success probability. We model uncertainty aversion by assuming outside investors maximize Minimum Expected Utility (MEU) as in Gilboa and Schmeilder (1989).

An important implication of uncertainty aversion is that probabilistic assessments (or "beliefs" in the sense of Savage, 1954, and de Finetti, 1974) held by an uncertainty-averse investor are not uniquely determined by a single prior but, rather, are determined endogenously as the outcome of a minimization problem. A well-known feature of the MEU approach is that uncertainty averse agents typically hold more "pessimistic" beliefs than corresponding expected utility agents.[2] In this paper, we exploit another important property of the MEU approach. Specifically, in our model, uncertainty-averse investors hold (weakly) more favorable probabilistic assessments toward an innovation, and thus value it more, if they invest in other innovations as well. This happens because, by holding a portfolio of uncertain assets, investors can lower their exposure to the source of uncertainty for each asset, a property known as uncertainty hedging.[3] The effect of uncertainty hedging is to lead investors to hold more favorable beliefs, and thus to be more "optimistic" on an asset's future profitability, when they hold multiple assets in their portfolio rather than in isolation, partially alleviating their "pessimism" relative to uncertainty-neutral investors.

In our economy, entrepreneurs raise equity from uncertainty-averse investors in the public equity market through an IPO. The key feature of our model is that uncertainty-averse investors, by investing in a portfolio of (possibly independent) R&D processes, reduce their exposure to the joint (uncertain) event that all such R&D efforts fail.[4] Thus, uncertainty-averse investors are (relatively) more optimistic, and willing to pay more, for equity in a given entrepreneurial firm when other entrepreneurs innovate as well. We refer to the (endogenous) probabilistic assessments held by investors on the success of innovations as characterizing their sentiment.

[2] This feature is used in Garlappi, Giammarino, and Lazrak (2017) and Lee and Rajan (2018), among others.

[3] Uncertainty hedging is a direct consequence of the "uncertainty aversion axiom" of Gilboa and Schmeidler (1989).

[4] For example, there is considerable uncertainty on the technical difficulties related to development of self-driving cars – where several companies are engaged in substantial R&D effort. Clearly, there is very little information on the true odds of discovery relevant for each producer. Most likely, however, one of such innovators will discover a workable technology that will become the industry standard. By investing in a portfolio of companies, investors (such as VCs) limit their exposure to the adverse effect of uncertainty, reducing their assessment of the probability each technology will fail and, thus, improving their overall outlook on the sector.

We show uncertainty aversion generates strategic complementarity between innovative activities which can result in innovation waves. Arrival of innovation opportunities in the economy may be the random effect of exogenous technological progress. We argue such technological advances, while seeding the ground for an innovation wave, may not be sufficient to ignite one. Rather, a wave will start when a critical mass of innovators is attained, which spurs a "hot" market for innovation.

The notion of sentiment we propose in our paper is firmly grounded on decision theory and provides a new channel for how it affects economic behavior. Several authors suggested that investor attitude toward investment in risky assets, often referred to as sentiment, is a critical feature of investor behavior in financial markets.[5] Unfortunately, there is still little understanding of its economic drivers and time variation. Our approach provides a foundation for the notion of sentiment based on uncertainty and investor aversion to it. In our model, investors beliefs are determined endogenously and, because of uncertainty hedging, depend on their changing overall exposure to the sources of uncertainty in the economy. Importantly, in our theory investor sentiment is not based on erroneous beliefs disjoint from economic fundamentals but, rather, it depends directly on investor uncertainty about such fundamentals.[6]

The channel we propose differs from more traditional "neoclassical" explanations. Shleifer (1986) argues that innovations in one sector have a positive externality in other sectors, because of their positive effect on aggregate demand. As in our paper, innovators prefer to postpone innovation to periods of time when other innovators undertake theirs, generating self-fulfilling boom-and-bust cycles. While in Shleifer (1986) this cycle occurs through the effect that a favorable aggregate macroeconomic environment has on the value of innovation, in our model waves may be localized in a specific sector, even if the overall economy is not booming. Thus, our model can explain the boom in the biotech market in 1989-1992, which occurred around the economic recession of thr early 1990's and within a relatively calmer overall stock market (Booth, 2016).

Acemoglu and Zilibotti (1997) argue that at the early stages of economic development, when

[5] See Barberis, Shleifer, and Vishny (1998), Baker and Wurgler (2006), and Ljungqvist, Nanda, and Singh (2006). Baker and Wurgler (2007) suggest investor sentiment, in the form of "optimism or pessimism about stocks," is likely to affect more those stocks that are harder to value, that is, stocks surrounded by more uncertainty.

[6] Our paper is related to recent rational explanations of sentiment, such as Angeletos and La'O (2013). An important difference with our paper is that in Angeletos and La'O a change in sentiment is the outcome of an aggregate shock (a "sentiment shock") that affects agent information sets and, thus, beliefs in the presence of imperfect communication. In our paper, changes in sentiment emerge endogenously in an otherwise stationary environment.

capital is limited, the presence of project indivisibilities caps the range of risky investment projects implemented in an economy, reducing the benefits of risk sharing, discouraging investment. Our paper differs in many important dimensions. First, our results do not rely on the limited supply of capital but, rather, are driven by the random arrival of innovative ideas in the economy. In our model capital is abundant and, thus, it is better suited to explain innovation waves in more mature economies, while the Acemoglu and Zilibotti model is better suited to explain the random growth rates of economies at the earlier stages of their development. Second, in Acemoglu and Zilibotti a "wave" (or, perhaps a "crash") may occur as the outcome of negative production shocks that reduce capital available in the economy thus restricting its diversification opportunities. In contrast, in our paper, a wave ends when the pipeline of innovations that were initiated in that wave are completed, and a new wave starts when a new critical mass is achieved.[7]

The benefits of uncertainty hedging are similar to the traditional benefits of risk diversification in standard portfolio theory, but in the context of uncertainty.[8] A key difference is that, under uncertainty aversion, investors' beliefs and attitude toward risky assets are endogenous and depend on their overall portfolio composition. In addition, because of uncertainty hedging, investors hold more favorable beliefs on the future cash flow of risky assets when they hold such assets in a portfolio rather than in isolation. In contrast, under traditional risk aversion, investors' subjective beliefs are fixed, and are uniquely determined by their (single) prior.

A second important difference is that traditional portfolio diversification can generate innovation waves coupled with high stock market valuations as the outcome of a reduction of the economy-wide market price of risk, as in Acemoglu and Zilibotti (1997). This channel requires a volume of new firms seeking financing that is sufficiently large to reduce the aggregate market risk premium, and innovation waves are necessarily associated with economy-wide equity market booms.

Our approach can explain the apparent "boom and bust" behavior in technology sectors, such as the Life Sciences and the Information Technology, where hot periods in innovation rates, merger activity, and asset valuations alternate with cold periods, even while the overall equity market remains stable. This divergence between a technology sector and the general market is difficult to

[7] Equilibrium investment booms and busts in the presence of asymmetric information and agency conflicts is also examined by Greenwald et al. (1984), Zeira (1999), and Kumar and Langberg (2013).

[8] Observational equivalence between models based on uncertainty aversion and those based on standard risk-averse models is an issue discussed in the literature (see, for example, Maenhout, 2004, and Skiadas, 2003, among others).

reconcile on the basis of risk aversion: early-stage innovations typically represent a small portion of the overall market, thus providing only limited diversification opportunity to investors. Further, if the risk faced by investors is essentially technological, with little or no correlation with aggregate risk (as in Pastor and Veronesi, 2009), risk premia cannot play a role in innovation process. Traditional risk aversion can generate waves and hot equity markets limited to specific sectors only in the presence of capital markets segmentations that reduce risk-diversification opportunities available to investors.[9] In contrast, uncertainty hedging can play a significant role even in the case of small uncertain investments and trigger waves limited to specific sectors. In our model, periods of strong innovative activity are accompanied by high valuations because innovation waves are, in equilibrium, associated with more optimistic expectations on future cash flows.

Our paper has implications for the impact of M&A activity and corporate ownership structure on innovation. In the new channel we propose, mergers of innovative firms create synergies and spur innovation. Positive synergies are endogenous, the direct outcome of the beneficial spillover on the probabilistic assessments of future returns on innovation due to uncertainty aversion. Our model also predicts that merger activities involving innovative firms are associated with more optimistic investor expectations and greater valuations. Thus, our paper can explain the strong M&A activity of firms that just performed an IPO documented in Celikyurt et al. (2010).[10]

Our paper contributes insights from uncertainty aversion to three separate strands of literature. First, and foremost, our paper belongs to the rapidly expanding literature on determinants of innovation waves.[11] In early research, which focused mostly on the technological "fundamentals" behind innovation, waves are driven by a technological breakthrough that affects an entire sector, with positive spillover across different technologies. More recently, Zeira (1999) shows that such waves may be amplified by information overshooting in the context of rational learning about changing fundamentals. Grenadier (1996) argues overinvestment within booms and subsequent busts can be the outcome of optimal dynamic exercise of real investment options.

More recent research focuses on the link between innovation waves, availability of financing, and stock market booms. Greenwald et al. (1984) argue that credit constraints can generate boom

[9] Section 6 discusses the effect of uncertainty aversion on investor portfolio formation, such as venture capitalists.
[10] Hart and Holmstrom (2010) develop a model where mergers create value by internalizing externalities.
[11] The critical role of innovation and innovation waves has been extensively studied at least since Schumpeter (1939) and (1942), Kuznets (1940), and, more recently, Aghion and Howitt (1992), and Klepper (1996).

and busts in investment, but do not directly focus on innovation. Scharfstein and Stein (1990) suggest reputation considerations by investment managers may induce them to herd, facilitating the financing of technology firms.[12] Ljungqvist, Nanda, and Singh (2006) argue that firms time their IPOs in periods of (exogenously determined) strong sentiment from retail investors. Gompers et al. (2008) show that periods of high stock market valuations are also associated with greater fund raising by VCs. Kumar and Langberg (2013) suggests that booms and busts in firm investment may be the outcome of agency conflict between shareholders and managers, a consideration absent in our paper. Nanda and Rhodes-Kropf (2013) find that in "hot markets" VCs invest in riskier and more innovative firms. Nanda and Rhodes-Kropf (2017) argue favorable financial market conditions reduce refinancing risk for VCs, promoting investment in more innovative projects. A positive effect of investor sentiment on innovation is documented in Aramonte and Carl (2018).

The second stream of literature is the recent debate on the links between technological innovation and stock market prices. Nicholas (2008) shows that an important driver of the stock market run-up experienced in the American economy in the late 1920's was the strong innovative activity by industrial companies which affected the market valuation of "knowledge assets." Two closely related papers are Pastor and Veronesi (2005) and (2009). The first paper argues that IPO waves can be the outcome of a change in the "fundamentals" characterizing a firm and its environment, such as an exogenous decrease in the market expected return. In our paper, in contrast, IPO waves occur in a stationary environment. The second paper argues stock market booms (and subsequent crashes) are the outcome of the changing nature of risk that characterizes technological revolutions, from idiosyncratic to systematic, and its impact on discount rates. In our model, periods of strong innovative activity are accompanied by high valuations because innovation waves are, in equilibrium, associated with more optimistic expectations on cash flows. Thus, our model, which focuses on expected cash flows, complements theirs, that focus on discount rates.

The third stream of literature focuses on the drivers of mergers and the impact of M&A activity on incentives to innovate. High stock market valuations are associated with M&A activity (Maksimovic and Phillips, 2001, and Jovanovic and Rousseau, 2001). Rhodes-Kropf and Viswanathan (2004) argue this is the outcome of misvaluation of the true synergies created in a merger when

[12] Gompers and Lerner (2000) find higher venture capital valuations are not linked to better success rates of portfolio companies. Perez (2002) shows technological revolutions are associated with "overheated" financial markets.

the overall market is overvalued. The impact of M&A activity on corporate innovative activity has been documented by several empirical studies. Phillips and Zhdanov (2013) show that a firm's R&D expenditures increase in periods of strong M&A activity in the same industry. Bena and Li (2014) argue that the presence of technological overlap between two firms' innovative activities is a predictor of the probability of a merger.[13] We study a parsimonious model based on uncertainty aversion proposing a novel direct link between stock price valuations, M&A activity, and innovation rates. In addition, investors' desire for uncertainty hedging creates the externality that, in our model, is at the heart of endogenous synergies in mergers of innovative companies.

The paper is organized as follows. In Section 1, we introduce the basic model and in Section 2, we derive the main results. Section 3 develops the dynamic model. Section 4 discusses the impact of competition. Section 5 examines the impact of mergers on incentives to innovate. Section 6 presents the main empirical implications of our model. Section 7 discusses certain critical assumptions and possible extensions of our model and concludes the paper. All proofs are in the Appendix.

1 The Basic Model

We study a two-period economy with three dates, $t \in \{1, 2, 3\}$. The economy has two classes of agents: investors and entrepreneurs. Entrepreneurs are endowed with project-ideas that may lead to an innovation. Project-ideas are risky and require an investment both at the beginning, $t = 1$, and at the interim date, $t = 2$. If successful, project-ideas generate a valuable innovation at $t = 3$ and, if unsuccessful, have zero payoff. For simplicity, we assume initially that there are only two entrepreneurs, denominated by τ, with $\tau \in \{A, B\}$. Project-ideas are creative innovations that are unique to each entrepreneur and can be pursued only by the entrepreneur generating them.

Entrepreneurs are penniless and require financing from investors. There is a unit mass of investors, each with ω_0 units of endowment. This initial endowment can be used to invest in one (or both) of the two project-ideas or in another (risky) asset that is available in the economy. Investment in the other asset, which can be interpreted as the market portfolio, can be made at

[13] Bernstein (2015) documents that in the three years after their IPO, firms engage in strong M&A activity, acquiring a substantial number of patents. Sevilir and Tian (2012) show that acquiring innovative target firms is positively related to acquirer abnormal announcement returns and long-term stock return performance. The importance of the presence of technological overlaps between acquiring firms and targets is confirmed by Seru (2014), which finds that innovation rates are lower in diversifying mergers, where the technological benefits of a merger are likely to be absent.

either $t = 1$ and $t = 2$, and yields a unit gross expected return per period (a normalization).

The innovation process is structured in two stages. To implement a project-idea, and thus to "innovate," an entrepreneur makes at $t = 1$ a fixed, non-pecuniary investment k_τ. This investment represents the preliminary personal effort (or "sweat equity") that she must exert to generate the idea. We refer to the initial investment, k_τ, as the "discovery cost" for the innovation. We denote the decision made by entrepreneur τ of whether to incur such cost with $d_\tau \in \{0, 1\}$, where $d_\tau = 1$ indicates the personal investment is made, and $d_\tau = 0$ otherwise. The innovation process is inherently risky: denote with q_τ the success probability of the first stage of the process. For simplicity, assume first-stage success probabilities are independent.[14]

If the first stage is successful, at $t = 2$ the innovation process enters the second stage. The second stage requires a monetary investment of c_τ. Entrepreneurs pay for this investment by selling equity to a large number of well-diversified investors in an IPO. Entrepreneurs are impatient and sell at the interim period, $t = 2$, their entire firm to outside investors, at (market) value V_τ. The second stage of the innovation process is also risky and, if successful, the innovation generates at the end of the last period, $t = 3$, the payoff y_τ with probability p, and zero otherwise. We also assume that the success probabilities of the second stage are independent. Finally, if an entrepreneur does not initiate the innovation process, she will have a zero payoff.

The game unfolds as follows. At the beginning, $t = 1$, entrepreneurs simultaneously decide whether or not to innovate, and select $d_\tau \in \{0, 1\}$. Investors invest their endowment in the other asset available in the economy. At the interim date, $t = 2$, entrepreneurs with successful first stage sell their entire firm to outside investors for value V_τ, which represents their payoff from the innovation. Outside investors purchase a fraction ω_τ of firm τ and invest the residual value $\omega_0 - \omega_A V_A - \omega_B V_B$ in the other asset. At the last stage, $t = 3$, residual uncertainty on the success of each innovation is resolved and payoff realized. Investors' final payoff depends on their investments in each innovation, ω_τ, and on the return from those investments. We initially assume that project payoffs are the same whether only one or both entrepreneurs are successful.

[14] Positive correlation of first-stage success probabilities would increase potential for waves.

1.1 Modeling Uncertainty

A key feature of our model is that outside investors are uncertain about the success probability of the second stage of project-ideas, p. We model uncertainty (or "ambiguity") aversion by adopting the Gilboa and Schmeidler (1989) Minimum Expected Utility approach:[15] economic agents do not have a single prior on future events but, rather, believe that the probability distribution of future events belongs to a given set $\mathcal{M}$, denoted as the "core beliefs set," and maximize

$$\mathcal{U} \equiv \min_{\mu \in \mathcal{M}} E_{\mu}\left[u\left(\cdot\right)\right], \tag{1}$$

where μ is a probability distribution, and $u\left(\cdot\right)$ is a von-Neumann Morgenstern utility function. An important property of uncertainty aversion that plays a critical role in our paper is that uncertainty-averse agents weakly prefer randomizations over random variables (more precisely, over acts described in Anscombe and Aumann, 1963) rather than each individual variable in isolation. This property is a direct consequence of the uncertainty-aversion axiom of Gilboa and Schmeidler (1989) and is known as "uncertainty hedging." Given two random variables, y_k, $k \in \{1, 2\}$, with joint distribution $\mu \in \mathcal{M}$, by the property of the minimum operator, for any $q \in [0, 1]$,

$$q \min_{\mu \in \mathcal{M}} E_{\mu}\left[u\left(y_1\right)\right] + (1-q) \min_{\mu \in \mathcal{M}} E_{\mu}\left[u\left(y_2\right)\right] \leq \min_{\mu \in \mathcal{M}} \{qE_{\mu}\left[u\left(y_1\right)\right] + (1-q)E_{\mu}\left[u\left(y_2\right)\right]\}. \tag{2}$$

The key driver of our results is that (2) can hold with strict inequality.

Investors are uncertain on the success probability of the second stage of the innovation, p. Following Hansen and Sargent (2001) and (2008) we characterize the core beliefs set $\mathcal{M}$ in (1) by using the notion of relative entropy.[16] For a given pair of (discrete) probability distributions $(p, \hat{p})$, the *relative entropy* of p with respect to $\hat{p}$ is the Kullback-Leibler divergence of p from $\hat{p}$:

$$R(p|\hat{p}) \equiv \sum_i p_i \log \frac{p_i}{\hat{p}_i}. \tag{3}$$

[15] An alternative approach is "smooth ambiguity" developed by Klibanoff, Marinacci, and Mukerji (2005). In their model, agents maximize expected felicity of expected utility, and agents are uncertainty averse if the felicity function is concave. The main results of our paper will hold in this approach (if the felicity function is sufficiently concave), at the cost of substantially greater analytical complexity. Our results also hold under variational preferences of Maccheroni, Marinacci, and Rustichini (2006) if the ambiguity index $c\left(p\right)$ has a positive cross-partial. See also Siniscalchi (2011).

[16] This specification of ambiguity aversion, which is often referred to as the "constrained preferences" approach, is a particular case of the larger class of "variational preferences." Strzalecki (2011) provides a general characterization of different approaches to modeling ambiguity aversion.

Thus, the core beliefs set for the uncertainty-averse investors in our economy is

$$\mathcal{M} \equiv \{p : R(p|\hat{p}) \leq \tilde{\eta}\}, \tag{4}$$

where p is the joint distribution of the success probability of the second stage of the two projects, and $\hat{p}$ is an exogenously given "reference" probability distribution. From (3), it is easy to see that the relative entropy of p with respect to $\hat{p}$ represents the (expected) log-likelihood ratio of the pairs of distributions $(p, \hat{p})$, when the "true" probability distribution is p. Thus, the core beliefs set $\mathcal{M}$ can be interpreted as the set of probability distributions, p, that, if true, the investor would expect not to reject the ("null") hypothesis $\hat{p}$ in a likelihood-ratio test.

Intuitively, the core belief set $\mathcal{M}$ includes probability distributions that are not "too unlikely" to be the true (joint) probability distribution that characterizes the two technologies, given the reference distribution $\hat{p}$. Thus, the reference distribution $\hat{p}$ can be interpreted as characterizing an agent's "view" about the true success probability p, and the parameter $\tilde{\eta}$ represents the degree of the agent's confidence on the reference probability.[17] A small value of $\tilde{\eta}$ represents situations where agents have more confidence that the probability distribution $\hat{p}$ is a good representation of the success probability of the two technologies, while a large value of $\tilde{\eta}$ corresponds to situations where there is great uncertainty on such probabilities.

An important effect of restricting investors' beliefs to the core beliefs set (4) is to rule out probability distributions that are "too far" from the reference probability $\hat{p}$. In other words, the maximum entropy criterion implied by (4) has the effect of excluding from the core-belief set probability distributions that give too much weight to extreme events (in both the right and left tails). Because uncertainty-averse investors are essentially concerned about "left-tail" events, we interpret this property as "trimming pessimism."[18]

Lemma 1 *Let $\tilde{\eta} < \tilde{\eta}^0(\hat{p})$ (defined in the appendix). The core beliefs set $\mathcal{M}$ is a strictly convex set with smooth boundary. Furthermore, if investors have nonnegative investments in both innovations, the solution to (1) is on the lower left-hand boundary of $\mathcal{M}$.*

[17] As in Hansen and Sargent (2001) and (2008), relative entropy characterizes extent of "misspecification error."

[18] Referring back to our example on self-driving cars, the relative entropy criterion eliminates from $\mathcal{M}$ probability distributions that give too much weight to the extreme event that *all* technologies under development will fail.

Lemma 1 is a direct implication of the strict convexity of relative entropy, $R(p|\hat{p})$.[19] Strict convexity of the core beliefs set $\mathcal{M}$ implies (2) holds with strict inequality, and, thus, investors prefer to hold uncertain technologies in a portfolio. Lemma 1 also implies uncertainty-averse investors with positive investment in both project ideas select probability assessments that lie in the "lower-left" boundary of $\mathcal{M}$, so the relevant part of the set is a smooth, decreasing, and convex function. Thus, the deterioration of an assessment of success probability of one technology is met by a corresponding improvement of assessment of success probability of the other technology.[20]

Unfortunately, the level set of relative entropy for binomial distributions in (4) is not solvable in closed form, making the model analytically intractable. Therefore, similar to Dicks and Fulghieri (2019), we use a parametrization of the core belief that closely approximates (4). First, we assume the success probability $p(\theta_\tau)$, $\tau \in \{A, B\}$, depends on the value of an underlying parameter θ_τ, and is $p(\theta_\tau) = e^{\theta_\tau - \theta_M}$, where $\theta_\tau < \theta_M$. Uncertainty-averse agents treat the vector $\vec{\theta} \equiv (\theta_A, \theta_B)$ as ambiguous, and they believe $\overrightarrow{\theta}$ is near $\overrightarrow{\theta}^* = (\theta^*, \theta^*)$, giving $\hat{p} = p(\theta^*)$. For a given parameter combination $\vec{\theta}$, the second-stage success probabilities are independent.[21] Second, we describe the relevant portion of the boundary of $\mathcal{M}$ (characterized in Lemma 4) by using the L^1 norm:

$$C \equiv \left\{ \overrightarrow{\theta} : \sum_{\tau \in \{A,B\}} |\theta_\tau - \theta^*| \leq \eta \right\}, \tag{5}$$

where C now denotes the "core beliefs" set. Because investors hold innovations with non-negative weights in their portfolios, we have $\theta_\tau \leq \theta^*$, which implies $\frac{1}{2}(\theta_A + \theta_B) = \theta^* - \frac{\eta}{2}$.

The relationship between the core belief set under the relative entropy criterion (4), $\mathcal{M}$, and the approximation of its lower left boundary under the L^1 norm (5), is depicted in Figure 1, drawn on the basis of the parameter specification displayed in Point 1 of Table 1. The table displays a numerical example that will illustrate the various cases we analyze in our paper. The figure displays the full core belief set $\mathcal{M}$ and, in bold, the lower-left boundary portion of $\mathcal{M}$ that is relevant for investors with long positions in both assets. The approximation of the lower-left boundary of $\mathcal{M}$

[19] For a general discussion, see Theorem 2.5.3 and 2.7.2 of Cover and Thomas (2006). Our results hold, generally, when the core belief set $\mathcal{M}$ is a strictly convex set with smooth boundaries. Note that "rectangular" core-belief sets do not satisfy such condition, thus defeating the benefits of uncertainty hedging of the uncertainty-aversion axiom.

[20] This holds because relative entropy is additively separable in independent variables.

[21] It is easy, although messy, to allow for the possibility of correlated second-stage success probabilities.

under the L^1 norm (5), is depicted with a dashed blue line.

Finally, while outside investors are uncertain on the parameter combination $\vec{\theta}$, there are no other sources of uncertainty in the economy and investors are otherwise risk-neutral.[22] This means investors can have access to other (risky) investment opportunities without affecting our results. We make this risk-neutrality assumption to isolate the effect of uncertainty aversion from the traditional (and well-understood) risk-aversion channel. For simplicity, we also assume entrepreneurs are both uncertainty and risk neutral. In Section 7 we discuss more explicitly the impact of these assumptions on our analysis. We will at times benchmark the behavior of uncertainty-averse agents with the behavior of an uncertainty-neutral SEU agent for whom $\eta = 0$, so that he has a prior belief $\theta_N = \theta^*$.

1.2 Uncertainty Aversion and Investor Sentiment

An important implication of uncertainty aversion is that the probability assessment (i.e., the "beliefs") held by an uncertainty-averse investor (that is, their assessment of the parameter combination $\vec{\theta}$) are endogenous, and depend on the composition of their overall portfolio. Endogeneity of beliefs is the outcome of the fact that, when the uncertainty-hedging condition (2) holds as a strict inequality, the minimization operator in (1) depends the agent's overall exposure to uncertainty.

The effect of uncertainty hedging is that investors hold more favorable probability assessments on the success probability of project-ideas if they invest in both projects, rather than in just one project. Specifically, if both innovations are successful, and an investor decides to purchase a proportion ω_τ of entrepreneur τ's firm, with payoff y_τ, the investor will hold a risky portfolio $\Pi = \{\omega_A y_A, \omega_B y_B, \omega_0 - \omega_A V_A - \omega_B V_B\}$. Because investors are uncertainty averse but otherwise risk neutral, portfolio Π provides the investor with utility $U(\Pi) = \min_{\overrightarrow{\theta} \in C} u\left(\Pi, \overrightarrow{\theta}\right)$, where

$$u\left(\Pi, \overrightarrow{\theta}\right) \equiv e^{\theta_A - \theta_M} \omega_A y_A + e^{\theta_B - \theta_M} \omega_B y_B + \omega_0 - \omega_A V_A - \omega_B V_B. \tag{6}$$

Because of uncertainty aversion, the investor's assessment at $t = 2$ on the state of the economy,

[22] Alternatively, our paper can be interpreted as modeling situations where uncertainty on other sectors of the economy does not affect uncertainty surrounding our firms.

$\overrightarrow{\theta}^a$, is the solution to the minimization problem

$$\overrightarrow{\theta}^a (\Pi) \equiv \arg\min_{\theta \in C} u\left(\Pi, \overrightarrow{\theta}\right). \tag{7}$$

Lemma 2 *Uncertainty-averse investors hold less favorable beliefs than uncertainty-neutral investors. Increasing an investor's exposure to one innovation induces a more favorable assessment of the other innovation. Formally, given portfolio* Π, *and letting*

$$\check{\theta}^a_\tau (\Pi) = \theta^* - \frac{\eta}{2} + \frac{1}{2} \ln \frac{\omega_{\tau'} y_{\tau'}}{\omega_\tau y_\tau}, \tag{8}$$

an uncertainty-averse investor holds an assessment θ^a_τ *on the uncertain parameter* θ_τ *equal to*

$$\theta^a_\tau (\Pi) = \begin{cases} \theta^* - \eta & \check{\theta}^a_\tau (\Pi) \leq \theta^* - \eta \\ \check{\theta}^a_\tau (\Pi) & \check{\theta}^a_\tau (\Pi) \in (\theta^* - \eta, \theta^*) \\ \theta^* & \check{\theta}^a_\tau (\Pi) \geq \theta^* \end{cases}. \tag{9}$$

If the investor invests in only one project idea, $\theta^a_\tau (\Pi) = \theta^* - \eta$.

Lemma 2 shows that uncertainty-averse investors' assessment on $\overrightarrow{\theta}$ is endogenous and determines his view on the success probability of the innovations, given their assessment of the degree of uncertainty surrounding the two firms. Thus, we refer to the assessment $\overrightarrow{\theta}^a \equiv (\theta^a_A, \theta^a_B)$, and the corresponding probabilistic assessments $\{p(\theta^a_A), p(\theta^a_B)\}$, as characterizing investor sentiment. Uncertainty-averse investors hold less favorable beliefs that uncertainty-neutral investors $p(\theta^a_\tau) \leq p(\theta^*_\tau)$, where their assessment $p(\theta^a_\tau)$ is a decreasing function of their degree of confidence, η, in the reference probability, $p(\theta^*_\tau)$. Investors' beliefs and, thus, their attitude toward risky assets, depend on their overall portfolio composition, Π, and the degree of confidence in the reference probability, η, due to uncertainty aversion. Critically, because of uncertainty hedging, investors hold more favorable beliefs on the return on risky assets when they hold such assets in a portfolio rather than in isolation. For ease of exposition, we assume project payoffs are not too different, $y_\tau / y_{\tau'} \in (e^{-\eta}, e^{\eta})$, so that $\theta^a_\tau (\Pi) \in (\theta^* - \eta, \theta^*)$ in (9).

From Lemma 2, when an investor has a relatively greater proportion of her portfolio invested in innovation τ, $\omega_\tau y_\tau > \omega_{\tau'} y_{\tau'}$, she will be relatively more pessimistic about the return on that innovation. This happens because a greater exposure to the uncertainty regarding a given innovation, relative to another innovation, will make an uncertainty-averse investor relatively more concerned about priors that are less favorable to that innovation. Correspondingly, the investor will give more weight to the states of nature that are more favorable to the other innovation. In other words, the investor will be more "optimistic" on the success probability of that other innovation.

A key implication of uncertainty hedging is to create a positive externality among investment projects through its effect on investors beliefs. Suppose entrepreneur τ has a successful first-stage project-idea, while entrepreneur τ' does not: Lemma 2 shows $\theta_{\tau}^{a}(\Pi) = \theta^{*} - \eta$ when $\omega_{\tau'} y_{\tau'} = 0$. In this case, at the interim date, $t = 2$, investors hold more pessimistic assessments about the successful innovation than if both entrepreneurs have a successful first-stage project-idea. By investing in only one project-idea, investors hold a portfolio with greater exposure to the possibility that the second-stage success probability is very low. In contrast, by investing in both technologies, investors limit exposure to the "tail event" that both project-ideas have a very low success probability in the second stage, a hypothesis rejected by the relative entropy criterion (4). Similar situations emerge if only one entrepreneur decides to innovate, while the other entrepreneur does not.

An important implication of Lemma 2 is that investor probabilistic beliefs about one innovation crucially depends on the availability of other innovations. Specifically, an investor will be more optimistic about an innovation (and, thus, values it more) if he will also be able to make greater investments in other innovations. This implies that uncertainty-averse investors perceive innovations effectively as complements and prefer to hold them in a portfolio.

2 Uncertainty and Innovation

Portfolio complementarity due to uncertainty aversion creates strategic complementarity among entrepreneurs. If only one entrepreneur innovates, she will face adverse investor beliefs. In contrast, if both entrepreneurs innovate and have a successful first stage, uncertainty-averse investors hold more favorable beliefs toward both innovations, making both innovations more valuable to the entrepreneurs. This implies that an entrepreneur is more willing to innovate if other entrepreneurs innovate as well. This positive spillover from one innovation to another generates innovation waves.

2.1 The Uncertainty-Neutral Case

As a benchmark, we start the analysis by characterizing innovation decisions when investors are uncertainty-neutral. If an entrepreneur has a successful first-stage project, under risk neutrality equity prices depend only on their prior $\theta_N = \theta^*$, giving

$$V_\tau \equiv p(\theta^*)\, y_\tau, \text{ for } \tau \in \{A, B\}, \tag{10}$$

Equation (10) shows that equity value for innovation τ depends only on its second-stage success probability, $p(\theta^*)$, and project payoff y_τ. The ex-ante expected payoff for entrepreneur τ from initiating the innovation process, and thus incurring discovery cost k_τ, is

$$E\mathcal{U}_\tau \equiv q_\tau \left[p(\theta^*)\, y_\tau - c_\tau\right] - k_\tau.$$

Entrepreneur τ innovates at $t = 0$ if $E\mathcal{U}_\tau \geq 0$, leading to the following theorem.

Theorem 1 *When investors are uncertainty-neutral, entrepreneurs of type τ innovate iff*

$$k_\tau \leq \bar{k}_\tau \equiv q_\tau \left[p(\theta^*)\, y_\tau - c_\tau\right], \quad \tau \in \{A, B\}.$$

Each entrepreneur initiates the innovation process when the expected payoff from the innovation exceeds the initial discovery cost. Importantly, when investors are uncertainty neutral, investment decisions by the two entrepreneurs are effectively independent, and they innovate only if their own discovery cost is not too large, $k_\tau \leq \bar{k}_\tau$. This threshold, $\bar{k}_\tau$, is increasing in the innovation potential payoff, y_τ, and both early-stage and late-stage success probabilities, q_τ and $p(\theta^*)$. Also, $\bar{k}_\tau$ is decreasing in the ex-post development cost, c_τ.

2.2 Uncertainty Aversion and Innovation

We solve the model recursively. First, we determine the value V_τ that investors are willing to pay at the interim date, $t = 2$, to entrepreneurs given first-stage success. Second, we solve for the initial choice by entrepreneurs to initiate the innovation process.

Interim-Date Equity Market Valuations. Implementation of the second stage of innovation requires entrepreneurs to raise capital by selling equity at $t = 2$. Entrepreneurs sell their entire firm to investors, use the proceeds to pay for the cost c_τ, and pocket the difference. The payoff to entrepreneur τ depends on the price outside investors are willing to pay, which, in turn, depends on their assessments on the success probability of the innovation, $p(\theta_\tau)$, as follows.

Theorem 2 *Uncertainty-averse (and risk-neutral) investors price entrepreneurial firms with successful first-stage projects at expected value, $V_\tau^a = p(\theta_\tau^a)y_\tau$, given their beliefs $\overrightarrow{\theta}^a$. If only one entrepreneur has successful first-stage project, firm value is*

$$V_\tau^a = p\left(\theta^* - \eta\right) y_\tau. \tag{11}$$

If both entrepreneurs are have successful first-stage projects, it is optimal for investors to hold a balanced portfolio, $\omega_A^* = \omega_B^*$, and firm values are

$$V_\tau^a = p\left(\theta^* - \frac{\eta}{2}\right) \left(y_\tau y_{\tau'}\right)^{1/2}, \ \textit{with} \ \ \tau, \tau' \in \{A, B\}, \tau \neq \tau'. \tag{12}$$

In addition, $V_\tau^a \leq V_\tau$, and V_τ^a is decreasing in η.

Theorem 2 shows that uncertainty-averse (and risk-neutral) investors price equity at its expected value, given their beliefs. If both firms are successful, investors hold a balanced portfolio by making equal investments in both firms. Desirability of equal investments is a consequence of uncertainty hedging. For simplicity, we normalize $\omega_A^* = \omega_B^* = 1$. Furthermore, because from Lemma 2 uncertainty-averse investors are less optimistic than uncertainty neutral investors, they have a lower equity valuation, $V_\tau^a \leq V_\tau$, where the uncertainty-aversion discount, $V_\tau - V_\tau^a$, is a increasing function of uncertainty, η.

Theorem 2 also shows that when investors are uncertainty averse, the market value of one firm depends not only on its own payoff, y_τ, but also on the one of other firm. The linkage between the market value of the two firms occurs through investor beliefs. From Lemma 2 an increase of the payoff of one firm will increase the relative exposure of investors to that firm's uncertainty relative to the other firm's uncertainty, making (all else equal) investors relatively more conservative (or pessimistic) about that firm's success probability and, correspondingly, relatively more confident (or optimistic) about the other firm's success probability. This interaction between equity market values of the two firms creates the strategic externality between the two entrepreneurs.

The market value of equity at the interim date depends on the number of firms seeking financing from investors, which in turn depends on whether only one or both firms have successful innovations. Thus, there are four possible states: (i) when both entrepreneurs had a successful first stage, state SS; (ii) when only one entrepreneur has a successful first-stage, state SF with the symmetric FS state, (iii) when both entrepreneur fail in the first stage and no innovation can take place, state FF. Since the last state FF is trivial, we focus on the first two.

Consider first the case in which only one entrepreneur had a successful first-stage project-idea, state SF. This state may arise either because the other entrepreneur has not initiated the innovation process or because the first stage was unsuccessful. Investors value equity under the worst case scenario giving $V_\tau^{a,SF} = p\left(\theta^* - \eta\right) y_\tau$, and the entrepreneur's continuation utility is

$$\mathcal{U}_\tau^{a,SF} \equiv V_\tau^{a,SF} - c_\tau = p\left(\theta^* - \eta\right) y_\tau - c_\tau. \tag{13}$$

If both entrepreneurs innovate and have a successful first-stage project-idea (state SS), equity market valuation is given in Theorem 2, with continuation utility

$$\mathcal{U}_\tau^{a,SS} \equiv V_\tau^{a,SS} - c_\tau = p\left(\theta^* - \frac{\eta}{2}\right)\left(y_\tau y_{\tau'}\right)^{1/2} - c_\tau. \tag{14}$$

From (8), the presence of two entrepreneurial firms makes investors more optimistic, leading to greater equity market valuations and entrepreneurial continuation utilities.

Corollary 1 *When both entrepreneurs have a successful first-stage project, they receive higher equity valuations, $V_\tau^{a,SS} > V_\tau^{a,SF}$, and are better off, $\mathcal{U}_\tau^{a,SS} > \mathcal{U}_\tau^{a,SF}$.*

The Innovation Decision. At $t = 1$, entrepreneurs must decide whether to sustain the (non-pecuniary) discovery cost k_τ to initiate innovation. If entrepreneur τ' chooses to innovate, $d_{\tau'}^a = 1$, the expected utility for entrepreneur τ from sustaining at $t = 1$ the initial discover cost k_τ is

$$E\mathcal{U}_\tau^{a,I} \equiv q_\tau\left[q_{\tau'}\mathcal{U}_\tau^{a,SS} + \left(1 - q_{\tau'}\right)\mathcal{U}_\tau^{a,SF}\right] - k_\tau$$

for $\tau, \tau' \in \{A, B\}$ and $\tau \neq \tau'$. Conversely, if entrepreneur τ' does not innovate at $t = 0$, $d_{\tau'}^a = 0$, the expected utility for entrepreneur τ from choosing to innovate at $t = 1$ is

$$E\mathcal{U}_{\tau}^{a,N} = q_{\tau}\mathcal{U}_{\tau}^{a,SF} - k_{\tau}.$$

Thus, entrepreneur τ objective function at $t = 1$ is given by:

$$E\mathcal{U}_{\tau}^{a} = d_{\tau'}^{a} E\mathcal{U}_{\tau}^{a,I} + (1 - d_{\tau'}^{a}) E\mathcal{U}_{\tau}^{a,N}. \tag{15}$$

Entrepreneur τ earns EU_{τ}^{a} if she sets $d_{\tau}^{a} = 1$, but zero if she selects $d_{\tau}^{a} = 0$. We characterize the equilibrium of the game by adopting the notion of subgame-perfect Nash Equilibrium.

Definition 1 *A subgame-perfect Nash Equilibrium is a strategy combination $\{d_{\tau}^{a}\}$ and investor equity valuation V_{τ}^{a}, for $\tau \in \{A, B\}$, such that: (i) each entrepreneur $\tau \in \{A, B\}$ at $t = 1$ maximizes (15), given the other entrepreneur's optimal strategy and investor equity valuation V_{τ}^{a}; (ii) investor equity valuation, V_{τ}^{a}, for $\tau \in \{A, B\}$, is from (11) and (12), given entrepreneurs' optimal strategies.*

The innovation decisions at $t = 1$ are as follows.

Theorem 3 *There are thresholds $\underline{k}_{\tau}^{a} < \bar{k}_{\tau}^{a}$, defined in the Appendix, with $\bar{k}_{\tau}^{a} < \bar{k}_{\tau}$, such that:*
(i) for low levels of discover cost, $k_{\tau} \leq \underline{k}_{\tau}^{a}$, an entrepreneur always innovates, $d_{\tau}^{a} = 1$;
(ii) for high levels of discovery cost, $k_{\tau} \geq \bar{k}_{\tau}^{a}$, an entrepreneur never innovates, $d_{\tau}^{a} = 0$;
(iii) for intermediate levels of the discovery cost, $k_{\tau} \in \left(\underline{k}_{\tau}^{a}, \bar{k}_{\tau}^{a}\right)$, an entrepreneur innovates only if the other entrepreneur innovates as well, $d_{\tau}^{a} = d_{\tau'}^{a}$.

For very small levels of discovery costs, $k_{\tau} \leq \underline{k}_{\tau}^{a}$, it is a dominant strategy for an entrepreneur to innovate. For very large levels of discovery costs, $k_{\tau} \geq \bar{k}_{\tau}^{a}$, it is a dominant strategy for an entrepreneur not to innovate. For intermediate levels of discovery costs, $k_{\tau} \in \left(\underline{k}_{\tau}^{a}, \bar{k}_{\tau}^{a}\right)$, entrepreneur τ wishes to innovate only if the other entrepreneur innovates as well. If both entrepreneurs have intermediate levels of discovery costs, there are two subgame perfect equilibria: one where both entrepreneurs innovate, $d_{A}^{a} = d_{B}^{a} = 1$, and one where neither innovate, $d_{A}^{a} = d_{B}^{a} = 0$. The equilibrium where both entrepreneurs innovate Pareto-dominates the no-innovation equilibrium. Finally, because uncertainty-averse investors hold less optimistic beliefs that uncertainty-neutral ones (as shown in Lemma 2), entrepreneurs will be less prone to innovate, $\bar{k}_\tau^a < \bar{k}_\tau$.[23]

When both entrepreneurs have intermediate discovery costs, there are equilibria with and without innovation. In this case, entrepreneurs face a classic "assurance game," in which there is a Pareto-superior equilibrium, where both entrepreneurs innovate, yet also an inefficient, Pareto-inferior equilibrium, where neither entrepreneur innovates. Multiplicity of equilibria results from strategic complementarity created by uncertainty aversion: it is profitable for an entrepreneur to innovate only if she expects high valuations, which requires the other entrepreneur to innovate.

Corollary 2 *The threshold levels* $\left\{\bar{k}_\tau^a\right\}_{\tau\in\{A,B\}}$ *are increasing functions of* $q_\tau, q_{\tau'}, y_\tau$, *and* $y_{\tau'}$, *and the threshold levels* $\{\underline{k}_\tau^a\}_{\tau\in\{A,B\}}$ *are increasing functions of* q_τ *and* y_τ. *Both* $\left\{\bar{k}_\tau^a\right\}_{\tau\in\{A,B\}}$ *and* $\{\underline{k}_\tau^a\}_{\tau\in\{A,B\}}$ *are decreasing in* η.

Corollary 2 shows an increase in one entrepreneur's probability of success, q_τ, makes not only that entrepreneur, but also others, more willing to innovate. Further, an increase in payoff of innovation increases not only that entrepreneur's willingness to innovate, but also makes innovation more attractive to others. An increase in uncertainty, η, makes innovation less attractive.

3 Investor Sentiment and Innovation Waves

We now examine the effect of uncertainty on innovation waves in the context of a simple dynamic model. We consider a simple discrete-time, infinite-horizon, dynamic stochastic game, where $t \in \mathbb{T} \equiv \{1, 2, 3, ...\}$ denotes time. At each date $t \in \mathbb{T}$, a new project-idea arrives with a constant, exogenous probability π, where each project-idea is owned by a unique entrepreneur. Let $\mathcal{E}_t$ be the set of entrepreneurs endowed with a project-idea at any given time $t \in \mathbb{T}$, and let $\nu_t \equiv |\mathcal{E}_t|$ be the number of entrepreneurs endowed with a project-idea.

Different from the basic model, we now assume an entrepreneur with a project-idea can delay implementation to a future date. Waiting to implement the project-idea, however, is costly: entrepreneurs and investors are impatient and have discount factor δ, which is the same for both groups.

[23] The uncertainty-neutral and uncertainty-averse cases for our numerical example are displayed in point 2 of Table 1. Uncertainty aversion decreases equity valuations with respect to uncertainty-neutral case and shrinks the set of values of the initial discover costs k for which an innovation is undertaken.

For tractability, we now assume that (i) if implemented, the first stage of the innovation process is always successful, setting $q = 1$; and (ii) project ideas have the same payoff, y, and costs c and k.[24]

An entrepreneur endowed with a project-idea at time t must decide whether to implement the innovation immediately or to delay its implementation to the next period. We assume this decision is made simultaneously by the entrepreneurs endowed with a project-idea only after observing a public signal which, for simplicity, we assume to be perfectly informative on ν_t.[25]

If an entrepreneur decides to innovate at time t, she must pay at that time discovery cost k to implement the first stage of the innovation process. At $t + 1$, successful entrepreneurs proceed with the second stage by sustaining cost c, paid by selling equity to investors, as described below. Finally, project-ideas implemented at time t have at time $t+2$ a payoff y with probability p, and 0 otherwise, after which the entrepreneur exits from the economy. If a project-idea available at t is not implemented, it is carried over at the following period, $t + 1$, where she again faces the choice of implementing the project-idea at that time, or to further delay implementation.

We model uncertainty in a way similar to the basic model. The success probability of the second stage of a project implemented by entrepreneur n at date t is uncertain, depending again on the value of a parameter θ_{nt}, and is equal to $p(\theta_{nt}) = e^{\theta_{nt} - \theta_M}$. For simplicity, we assume that uncertainty on p is stationary and independent across time, setting $\theta_{nt} = \theta_n$ for all (n, t).[26] Thus, at any time t, investors are uncertain over $\overrightarrow{\theta}$, and believe that

$$\overrightarrow{\theta} \in C \equiv \left\{ \overrightarrow{\theta} : \sum_{n \in \mathcal{E}_t} |\theta_\tau - \theta^*| \leq \eta \right\} \tag{16}$$

for some θ^* and $\eta \geq 0$. In this representation, $\overrightarrow{\theta}^*$ is the vector of parameters that support the reference probability $\hat{p}$ in (4) and η is the perceived degree of uncertainty on $\overrightarrow{\theta}$.

We solve the model by examining first sub-games in which (some) entrepreneurs seek financing at time t, because they initiated the innovation process at time $t-1$. In this case, uncertainty-averse investors form at time t portfolios of uncertain assets by buying equity from available entrepreneurs. Denote by S_t the set of entrepreneurs seeking financing at t, and let $s_t \equiv |S_t|$. Given our assumption

[24] Our results could be extended to the case where $q < 1$ but at the cost of greatly complicating the analysis.
[25] It is possible, although messy, to extend the model to the case in which the public signal is noisy.
[26] This assumption rules out issues related to learning, which we leave for future research.

that all innovations undertaken by an entrepreneur have a successful first-stage, S_t is the set of entrepreneurs that initiated their project-ideas at time $t-1$.

Similar to the basic model, each investor chooses a portfolio of the uncertain assets, $\{\omega_{nt}\}_{n \in S_t}$, given their market valuations $\{V_{nt}\}_{n \in S_t}$. By identical reasoning as Theorem 2, investors optimally invest equally in all available innovations: $\omega_{nt} = \omega_{n't}$ for all $n, n' \in S_t$ and $t \in \mathbb{T}$. Further, given investor assessments $\overrightarrow{\theta}_t^a$ at $t \in \mathbb{T}$, equity is priced at expected value: $V_{nt}^a = \delta p\left(\theta_{nt}^a\right) y$ for all $n \in S_t$. We again interpret $\overrightarrow{\theta}_t^a$ as characterizing investor beliefs (that is, their sentiment) in the equity market at time t. Importantly, similar to the basic model, investor assessment depends on the number s_t of entrepreneurs that seek financing in the equity market at t.

Lemma 3 *If an entrepreneur develops innovation alone, $s_t = 1$, investors are pessimistic:*

$$V_{nt}^a(1) = \delta p(\theta^* - \eta) y. \tag{17}$$

If an entrepreneur develops innovation with others, $s_t > 1$, investors will be more optimistic

$$\theta_{nt}^a(s_t) = \theta^* - \frac{\eta}{s_t}. \tag{18}$$

Correspondingly, the market value of equity will be

$$V_{nt}^a(s_t) = \delta p\left(\theta^* - \frac{\eta}{s_t}\right) y, \tag{19}$$

where $V_{nt}^a(s_t)$ is increasing in s_t. If $s_t = 0$ the equity market is closed, and $V_{nt}^a = 0$.

Lemma 3 shows investor beliefs at date t depends on the number of entrepreneurs, s_t, who initiated the innovation process the previous period, $t-1$, and are actively seeking financing. When an entrepreneur innovates unilaterally, $s_t = 1$, investor sentiment is weak, $\theta_{nt} = \theta^* - \eta$, and the capital market values innovations conservatively, $V_n^a = \delta p(\theta^* - \eta) y$, generating a "cold market."

In contrast, when entrepreneurs innovate together, for $s_t > 1$, from (18) assessed probabilistic beliefs improve with the number of projects available at that time, s_t, leading to a "hot market." With a larger number of projects available, uncertainty-averse investors reduce their exposure to the uncertainty of each individual project available at that time. Reduced exposure to project uncertainty, in turn, leads uncertainty-averse investors to hold more optimistic beliefs for each project and, thus, to a hot equity market. Similarly, investor beliefs are negatively affected by the extent of uncertainty in the economy: a greater value of η increases the total uncertainty faced by investors, which makes them relatively more pessimistic and leads to lower equity valuations.

We can now focus on the equilibrium of the full game. We use the notion of Markov Perfect Equilibrium from Maskin and Tirole (2001). A strategy is a Markov Strategy if and only if it depends only on the current state of the game, which in our setting is the number of entrepreneurs present in the economy, ν_t. Thus, we let the development decision of entrepreneur n be $d_n^a(\nu_t) \in \{0,1\}$, and we focus on equilibria with symmetric pure Markov Strategies.

Payoffs for a given entrepreneur are determined as follows. If an entrepreneur develops her project at time t, $d_n^a(\nu_t) = 1$, she expects the total number of projects available to investors to be $s_t = 1 + \sum_{m \in \mathcal{E}_t \setminus \{n\}} d_m^a(\nu_t)$. Let the development decisions of entrepreneurs other than n be $d_{n-}^a(\nu_t)$. From Lemma 3, she will sell equity in her project next period for $V_{nt}^a(s_t)$, yielding utility at t equal to $u_n^a\left(\nu_t, 1, d_{n-}^a\right) = \hat{u}(s_t)$ where

$$\hat{u}(s_t) \equiv \delta\left[V_{nt}^a(s_t) - c\right] - k. \tag{20}$$

Alternatively, if the entrepreneur chooses not to develop her project at time t, $d_n^a(\nu_t) = 0$, she will earn 0 at t but will still have the project at time $t+1$. Delaying innovation has both a cost and a benefit. The cost of delaying the innovation is given by discount factor, δ, while the benefit is that a new entrepreneur may arrive in the economy, which will increase the expected market value of the equity of her company, (19). The overall utility can be represented recursively as

$$\mathcal{U}_n^a\left(\nu_t, d_{n-}^a\right) \equiv \max_{d_n^a \in \{0,1\}} \left\{d_n^a u_n^a\left(\nu_t, 1, d_{n^-}^a\right) + (1 - d_n^a)\,\delta E \mathcal{U}_n^a\left(\nu_{t+1}, d_{n^-}^a\right)\right\}. \tag{21}$$

Definition 2 *A Markov Perfect Equilibrium is a strategy combination $\{d_n^{a*}(\nu_t)\}_{n \in \mathcal{E}_t}$ such that each entrepreneur maximizes (21) and investors value equity according to (17) and (19).*

Given strategies of other entrepreneurs, d_{n-}^a, at each point in time each entrepreneur optimally decides whether to develop her innovation. Markov Perfect Equilibria for our game may or may not have innovation waves, where two or more entrepreneurs are innovating at the same date.

Theorem 4 *There are values $\{\underline{k}_d, \overline{k}_d\}$, defined in the Appendix, such that:*

(i) Equilibria with waves: for $k \in (\underline{k}_d, \overline{k}_d)$ there is a threshold $\nu^* \geq 2$ such that any Markov Perfect Equilibrium has $d_n^{a*}(\nu_t) = 0$ for $\nu_t < \nu^*$ and $d_n^{a*}(\nu^*) = 1$. When there are ν_t entrepreneurs with project-ideas, the corresponding equilibrium payoff is

$$\mathcal{U}_n^a(\nu_t, d_{n^-}^*) = \left[\frac{\delta\pi}{1 - \delta(1-\pi)}\right]^{\nu^* - \nu_t} \hat{u}(\nu^*). \tag{22}$$

(ii) No-waves equilibria: for $k < \underline{k}_d$ the unique Markov Perfect Equilibria has $d_n^{a*}(1) = 1$, where an entrepreneur starts an innovation as soon as one is available; if $k > \overline{k}_d$ the innovation is not viable and $d_n^{a*}(\nu_t) = 0$, for all ν_t, $t \geq 0$.

Innovation waves exist only for intermediate values of the initial discovery cost k. If this cost is sufficiently low, $k < \underline{k}_d$, entrepreneurs prefer innovating immediately rather than waiting for a wave with better equity prices. Conversely, if the initial discovery cost is sufficiently large, $k > \overline{k}_d$, initiating innovation is too costly, making the innovation unprofitable.

In an innovation wave, investor beliefs, market valuations of firm equity, and innovation decisions are all endogenous, and depend on the number of innovative firms available on the market. If too few entrepreneurs are endowed with a project-idea, from (18), they all rationally anticipate that in the following period investors will be pessimistic ("adverse sentiment"), and correspondingly, market valuations will be low. This expectation of "cold equity markets" induces entrepreneurs to delay innovation. When the number of entrepreneurs with an innovation is greater than a certain critical mass, ν^*, entrepreneurs anticipate that, if they all innovate, investors will hold more favorable beliefs in the following period, and, correspondingly, market valuations will be higher. The expectation of a "hot equity market" will thus induce entrepreneurs to innovate.

Theorem 5 *There is a $k_0 \in (\underline{k}_d, \overline{k}_d)$ such that: (i) if $k \in (\underline{k}_d, k_0)$, there is a smallest possible wave, $\underline{\nu} = 2$, and a largest possible wave, $\bar{\nu}$; (ii) if $k \in (k_0, \overline{k}_d)$, there is a smallest wave, $\underline{\nu} \geq 2$, but no largest wave; (iii) for $k \in (\underline{k}_d, \overline{k}_d)$, in the efficient Markov Perfect equilibrium entrepreneurs innovate as soon as their number exceed critical mass $\nu^e \geq \underline{\nu}$.*

When the initial discovery costs are relatively small, $k \in (\underline{k}_d, k_0)$, a wave of two entrepreneurs, $\underline{\nu} = 2$, is viable. In this case, an entrepreneur is willing to innovate even if she anticipates that only one other entrepreneur innovates as well. The largest possible wave size $\bar{\nu}$ is determined by the fact that, when ν^* is too large, the expected (discounted) payoff from waiting for a wave becomes sufficiently small that the entrepreneur prefers to innovate alone. When initial discovery costs are moderate, $k \in \left(k_0, \bar{k}_d\right)$, an innovation is not profitable when implemented unilaterally, and it becomes viable only if implemented in wave, even if it requires a long waiting period.[27]

The efficient wave size, ν^e, maximizes the ex-ante payoff of entrepreneurs. A greater number of entrepreneurs has a beneficial effect on investor beliefs, raising equity prices in the wave. However, larger waves require more time to build-up, entailing greater deferral costs. The first effect dominates for small numbers of entrepreneurs, while the second dominates for larger waves.

Corollary 3 *The following properties hold: (i) $\underline{k}_d$ is decreasing in $\{\eta, \pi\}$; (ii) $\underline{v}$ is increasing in $\{\eta, k\}$, and decreasing in δ; (iii) the efficient wave, ν^e, is increasing in $\{\eta, k, \pi\}$.*

Corollary 3 has the following implications. Entrepreneurs find it more attractive to wait for a wave, rather than innovating immediately, when there is more uncertainty ($\underline{k}_d$ decreasing in η). Also, it is more attractive to wait for a wave when the arrival rate of the innovation, π, is larger ($\underline{k}_d$ decreasing in π), because a greater π makes the wave come faster.

Greater uncertainty leads also to a larger efficient wave, ν^e. From (18), a greater value of η makes uncertainty-averse investors relatively more pessimistic, so a greater number of entrepreneurs is needed to improve investor beliefs sufficiently to ignite innovation. Similarly, greater discovery cost k requires more favorable investor beliefs, and thus greater valuations, to induce entrepreneurs to initiate the innovation process, increasing both $\underline{\nu}$ and ν^e. Finally, increasing arrival rate π makes waiting for a larger critical mass more attractive, increasing ν^e.

Our model has the following implications for the innovation process in an economy. For low innovation costs, $k < \underline{k}_d$, entrepreneurs initiate the innovation process immediately. For intermediate innovation costs, $k \in \left(\underline{k}_d, \overline{k}_d\right)$, innovation activity remains latent in the economy when the number of entrepreneurs with project-ideas is below critical mass. During this time, entrepreneurs delay innovation, the market for entrepreneurial equity is "cold," and dominated by investors' negative

[27] Point 3 of Table 1 displays the dynamic version of our numerical example with innovation waves. The minimum wave has $\underline{\nu} = 2$, the efficient wave has $\nu^e = 7$, and equity valuations when entrepreneurs innovate alone, $V_{nt}^a(1) = 7.70$, and when they wait for the wave, $V_{nt}^a(\nu^e) = 34.16$.

outlook. When the number of entrepreneurs with project-ideas reaches critical mass, entrepreneurs expect a substantial improvement in investor sentiment and a "hot" equity market for innovations. The improved expectations on the future market conditions spark an innovation wave that ripples through the economy. In addition, Corollary 3 implies that greater uncertainty, or a greater discovery cost, will lead to less frequent innovation waves, but when the wave takes place it will involve a larger number of innovations and will be characterized by improved investor beliefs and equity valuations. Alternatively, if we interpret η as characterizing the complexity of an industry, Corollary 3 implies that less complex industries are characterized by more frequent innovation waves, of smaller intensity, and with less ebullient equity markets. In contrast, more complex industries are characterized by relatively less frequent innovation waves but, when they occur, they are of greater intensity and with more ebullient equity markets.

4 Competition and Innovation

We have assumed so far that the payoff from an innovation, y_τ, is not affected by the number of successful innovators and, thus, is shielded from competition. Thus, innovators always benefit from the presence of more innovators, because they increase the market value of their innovation due to uncertainty hedging. Competition may mitigate our results if it decreases the value of innovation. In this section, we explicitly consider the effect of competitive spillovers.

We modify our model as follows. We assume the payoff at $t = 3$ of innovation depends on the number of entrepreneurs with successful first-stage projects: when there are s successful first-stage project ideas, the potential payoff at time $t = 3$ from an innovation is $(1-\xi)^{s-1}\, y$, where $\xi \in (0, 1)$. The reduction of payoff captures adverse effects of competitive pressure in both the downstream markets (i.e., competition for customers) and in the upstream markets for resources (such as for limited financial or human capital, or other project inputs markets, including labor).[28]

Further, we allow for the possibility that arrival of a new project-idea can render obsolete older project-ideas which arrived in earlier periods and have been postponed. This captures the notion that delaying initiation of a project-idea exposes the entrepreneur to the possibility of being leapfrogged by new emerging competitors. For tractability, we assume that with probability ε, all

[28] $\xi < 0$ would imply a beneficial network effect, a situation that we do not examine and leave for future research.

delayed projects become obsolete, and therefore have no value. To simplify exposition, we assume symmetry across entrepreneurs: $q_\tau = 1$, $y_\tau = y$, $c_\tau = c$, and $k_\tau = k$.

Competition affects the equilibrium of the dynamic game as follows. First, investor beliefs are not affected by competition, because all payoffs are harmed equally by the same factor ξ. From Lemma 3, entrepreneurs with successful first-stage projects sell them to external investors for

$$V_{nt}^{ac}(s) = \delta p \left(\theta^* - \frac{\eta}{s}\right)(1-\xi)^{s-1} y. \tag{23}$$

We will again focus on Markov Perfect Equilibria with innovation waves.

Theorem 6 *There are thresholds* $\{\underline{k}_d^c(\xi,\varepsilon), \bar{k}_d^c(\xi), \underline{\xi}(\varepsilon)\}$, *such that:*
(i) weak competition: if $\xi < \underline{\xi}(\varepsilon)$, *and* $k \in \left(\underline{k}_d^c(\xi,\varepsilon), \bar{k}_d^c(\xi)\right)$, *the Markov Perfect Equilibrium has* $d_n^{a*}(\nu_t) = 0$ *for* $\nu < \nu^{c*}$ *and* $d_n^{a*}(\nu^{c*}) = 1$, *where* $\nu^{c*} \geq 2$. *The efficient Markov Perfect equilibrium has entrepreneurs innovating as soon as their number exceed the critical mass* $\nu^e(\xi,\varepsilon)$*;*
(ii) strong competition: if $\xi \geq \underline{\xi}(\varepsilon)$, *then* $\underline{k}_d^c(\xi,\varepsilon) = \bar{k}_d^c(\xi)$ *and entrepreneurs innovate immediately or not at all: there are no waves;*
(iii) If $k < \underline{k}_d^c(\xi,\varepsilon)$, *the unique Markov Perfect Equilibria has* $d_n^{a*}(1) = 1$*: entrepreneurs start the innovation process as soon as one is available. If* $k > \overline{k}_d^c(\xi)$, *the innovation is not viable and* $d_n^{a*}(\nu_t) = 0$, *for all* ν_t, $t \geq 0$.
The threshold $\underline{k}_d^c(\xi,\varepsilon)$ *is increasing in* ξ *and* ε, *with* $\frac{\partial^2 \underline{k}_d^c}{\partial\xi\partial\eta} < 0$, *while* $\underline{\xi}(\varepsilon)$ *is decreasing in* ε.

Competitive pressure reduces the benefits of innovation waves, making waves overall relatively less attractive. When competition is relatively weak, $\xi < \underline{\xi}$, and the initial discovery cost, k, is moderate, waiting for a wave is optimal for an entrepreneur. When competition is severe, $\xi \geq \underline{\xi}$, $\underline{k}_d^c(\xi) = \bar{k}_d(\xi)$. Thus, when the initial discovery cost is sufficiently low, $k < \underline{k}_d^c(\xi)$, an entrepreneur starts the innovation process only if she is the only innovator, $\nu = 1$, and no wave will occur. In this case, an entrepreneur with an innovation prefers to preempt potential competition by initiating the innovation process immediately and, thus, to become effectively a (temporary) monopolist. This incentive to innovate alone, however, is mitigated by the presence of uncertainty. Indeed, if $\eta = 0$, we have $\underline{\xi} = 0$, and entrepreneurs strictly prefer innovating alone. Finally, an increase in the obsolescence probability, ε, makes waiting for innovation waves less desirable.

Competition affects both the size of efficient innovation waves, ν^e, and innovation rates. Let T^e be the expected time between efficient innovation waves, and define rate of innovation

$$R^e \equiv \frac{\nu^e}{T^e}. \tag{24}$$

where R^e represents the long-term average innovation rate per unit of time.

Corollary 4 *The efficient wave, $\nu^e(\xi, \varepsilon)$, is U-shaped function of ξ, and the innovation rate R^e is a inverse U-shaped function of ξ. An increase of ε decreases ν^e.*

The number of entrepreneurs (and thus competitors) has now two opposing effects on the value of innovation in a wave. As in the basic model, the presence of more innovating firms improves investor beliefs, with a positive effect on equity valuations. The sentiment effect, however, is dampened by the negative effect of competition on the final payoff from the innovation, with an adverse effect on equity valuations. The competition effect dominates when competition is relatively less severe (low ξ), making smaller waves more desirable. When competition becomes sufficiently severe, high ξ, the sentiment effect once again dominates, because a larger wave is necessary to make innovation feasible, leading to a U-shaped relation between wave size and competition. Correspondingly, larger innovation waves require more time to build up, lowering per-period innovation rates, thus generating an inverse U-shaped function (of innovation rates R with respect to competition, ξ).[29]

5 Innovation and Acquisitions

Innovation is inherently a dynamic process, where firms can make additional investments that increase the final payoff from their innovations. In this section, we allow entrepreneurs to change their investment in the second stage of the innovation process, c_τ, and thus affect the final project payoff, y_τ. We interpret this choice as the determination of the intensity, or scale, of the innovation process. Innovation intensity reflects, for example, the level of R&D expenditures committed to the innovation which affect the ultimate value of the innovation.

[29] Point 4 of Table 1 displays the dynamic version of our numerical example with innovation waves and competition. Competitive pressure reduces the efficient wave to $\nu^{c*} = 4$ and the corresponding on the wave equity valuations, $V^a_{nt}(\nu^{c*}) = 20.67$. The table also displays the expecetd time between effcient waves waves, T^e, and the corresponding innovation rate, R^e.

The possibility of making investments affecting the level of y_τ creates a new externality to the one discussed in Section 2. The additional externality is that, from (12), the market value of an individual firm, V^a_τ, is increasing in the payoffs of both firms, $\{y_\tau, y_{\tau'}\}$. If firms choose their level of innovation intensity individually, they ignore the impact of their choice on the other firm's valuation through investor beliefs. This externality creates strategic complementarity between the choices of innovation intensity, y_τ, triggering the possibility of value-increasing mergers.

5.1 The Choice of Innovation Intensity

We modify our basic model as follows. If the first stage is successful, at $t = 2$, and before the sale of equity to outside investors, the entrepreneur decides the level of intensity of the innovation process, y_τ. We assume again that the payoff from innovation is not affected by potential competition among entrepreneurs (which is not our main concern in this section), so we set $\xi = 0$. Innovation intensity is costly: entrepreneur τ implementing y_τ sustains cost $c_\tau(y_\tau) = \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$, where Z_τ represents the productivity of entrepreneur τ's project-idea, and $\gamma > 0$ determines the convexity of its cost structure. We assume both innovation intensity y_τ and related costs $c_\tau(y_\tau)$ are contractible, eliminating moral hazard concerns (which are not the focus of our paper).

In the benchmark case, where investors are uncertainty neutral, investors value new innovation at $V_\tau \equiv p(\theta^*)\, y_\tau$, which implies optimal innovation intensity of $y^*_\tau \equiv [p(\theta^*)\, Z_\tau]^{\frac{1}{\gamma}}$. Similar to the discussion in Section 2, both entrepreneurs decide whether or not to innovate independently, with no role for the other entrepreneur's characteristics, either ex-ante or ex-post.

When investors are uncertainty averse, the choice of innovation intensity depends on the degree of investor sentiment. To simplify the exposition, we make the following regularity assumption:

$$A1\text{:}\;\; \eta > 2\ln(2) \;\text{ and }\; \frac{Z_\tau}{Z_{\tau'}} \in \left(\frac{1}{\psi}, \psi\right), \;\text{ where } \psi = \left[\frac{1}{4}e^{\eta}\right]^{\gamma+1} > 1, \tag{25}$$

which guarantees $\theta^a_\tau(\Pi) = \tilde{\theta}^a_\tau(\Pi)$ in Lemma 2 and existence of pure-strategy equilibria. Assumption $A1$ is satisfied when there is sufficient degree of uncertainty and the two firms are not too dissimilar.

We start with the case where only one entrepreneur is successful, while the other has either failed or not attempted first-stage innovation, state SF. The entrepreneur chooses the level of innovation intensity, y_τ, anticipating that investors will value equity under the worst case scenario,

$p(\theta^* - \eta)$. Substituting firm value from Theorem 2, the entrepreneur solves

$$\max_{y_\tau} \quad \mathcal{U}_\tau^{a,SF} \equiv p(\theta^* - \eta) y_\tau - \frac{1}{(1+\gamma) Z_\tau} y_\tau^{1+\gamma}. \tag{26}$$

The entrepreneur chooses a level of innovation intensity that reflects negative market sentiment:

$$y_\tau^{a,SF} \equiv [p(\theta^* - \eta) Z_\tau]^{\frac{1}{\gamma}} < y_\tau^{*n}. \tag{27}$$

The market value of her firm and her corresponding continuation utility are

$$V_\tau^{a,SF} \equiv [p(\theta^* - \eta)]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \text{ and } \mathcal{U}_\tau^{a,SF} = V_\tau^{a,SF} \frac{\gamma}{1+\gamma}. \tag{28}$$

More interesting is the case when both first-stage projects are successful, state SS, and

$$\mathcal{U}_\tau^{a,SS} \equiv V_\tau^{a,SS}(y_\tau, y_{\tau'}) - c_\tau(y_\tau) = p(\theta_\tau^a) y_\tau - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}, \tag{29}$$

where $V_\tau^{a,SS}$ is from (12) in Theorem 2, given payoffs $\{y_\tau, y_{\tau'}\}$. The choice of innovation intensity by entrepreneur τ is now determined by three factors. The first two factors are the direct impact of innovation intensity on firm value, for given beliefs, and its impact on cost, $c(y_\tau)$; these factors are in common with the entrepreneur maximization problem in the SF state, (26). Uncertainty aversion introduces a third factor: increasing the innovation intensity, y_τ, induces investors to be more pessimistic about the ultimate success probability for that firm, $p(\theta_\tau^a)$, decreasing its value, $V_\tau^{a,SS}$, with an adverse effect on expected payoff of the innovation. The subgame-perfect Nash Equilibrium is characterized in the following.

Theorem 7 *If both entrepreneurs have a successful first-stage project-idea (state SS), they select innovation intensity according to*

$$Y_\tau^{a,SS}(y_{\tau'}) = \left[\frac{1}{2} p\left(\theta^* - \frac{\eta}{2}\right) Z_\tau (y_{\tau'})^{1/2}\right]^{\frac{1}{\gamma + \frac{1}{2}}}, \quad \text{with } \tau \neq \tau', \text{ and } \tau, \tau' \in \{A, B\}; \tag{30}$$

which is increasing in the other entrepreneurs's innovation intensity, $y_{\tau'}$. The subgame-perfect

Nash Equilibrium innovation intensities for the sub-game are

$$y_\tau^{a,SS} = \left[\frac{1}{2} p \left(\theta^* - \frac{\eta}{2}\right) Z_\tau^\chi Z_{\tau'}^{1-\chi}\right]^{\frac{1}{\gamma}}, \tag{31}$$

where $\chi \equiv \frac{2\gamma+1}{2(1+\gamma)}$. *The equilibrium market value,* $V_\tau^{a,SS}$*, of the entrepreneurial firms and the corresponding continuation utility are, respectively,*

$$V_\tau^{a,SS} = \frac{1}{2^\gamma}\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{1+\gamma}{\gamma}} (Z_\tau Z_{\tau'})^{\frac{1}{2\gamma}} \text{ and } \mathcal{U}_\tau^{a,SS} = \chi V_\tau^{a,SS}. \tag{32}$$

As shown in Section 2, investors treat the innovative projects as complements, leading to a strategic complementarity between the choices of innovation intensity levels, y_τ. In addition, more favorable investors' beliefs lead to greater equity market valuations and higher levels of innovation intensity by both entrepreneurs as established in the following corollary.

Corollary 5 *If both entrepreneurs are successful, they implement more innovation,* $y_\tau^{a,SS} > y_\tau^{a,SF}$*, receive higher valuations,* $V_\tau^{a,SS} > V_\tau^{a,SF}$*, and are better off,* $\mathcal{U}_\tau^{a,SS} > \mathcal{U}_\tau^{a,SF}$*.*

5.2 Mergers and Synergies

Different from the basic model, if both entrepreneurs are successful in the first stage, state SS, at the interim date, $t = 2$, they now have the option to merge. After the merger, they jointly determine the innovation intensity, y_τ, for both innovation processes $\tau \in \{A, B\}$, and then the merged firm sells its equity in the public equity market.

After the merger, the new firm maximizes the combined value of the innovation projects. By reasoning from Theorem 2, the merged firm values projects at $V_\tau = p(\theta_\tau^m) y_\tau$, for $\tau \in \{A, B\}$, where θ_τ^m is the investors' assessment when the merged firm is sold. Thus, the merged firm's objective is

$$\mathcal{U}^m \equiv p(\theta_A^m) y_A + p(\theta_B^m) y_B - c_A(y_A) - c_B(y_B).$$

If investors are uncertainty neutral, $\theta_\tau^m = \theta^*$, then the choice of y_A and y_B are again independent: the merged firm solves the same problem as the original entrepreneurs, leading to the same optimal levels of innovation intensity and $\mathcal{U}^m = \mathcal{U}_A + \mathcal{U}_B$. Thus, the merger does not add value.

If investors are uncertainty averse, $\overrightarrow{\theta}^m = \overrightarrow{\theta}^a$ from (9), which depends on both y_A and y_B. As in Theorem 2, $V_A = V_B = p\left(\theta^* - \frac{\eta}{2}\right) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}}$. The maximization problem of the merged firm becomes

$$\mathcal{U}^{a,m} = 2p\left(\theta^* - \frac{\eta}{2}\right) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}} - \frac{1}{Z_A(1+\gamma)} y_A^{1+\gamma} - \frac{1}{Z_B(1+\gamma)} y_B^{1+\gamma}.$$

Theorem 8 *When both entrepreneurs are successful, they merge and implement greater innovation intensity,* $y_\tau^{a,m} \equiv \left[p\left(\theta^* - \frac{\eta}{2}\right) Z_\tau^\chi Z_{\tau'}^{1-\chi}\right]^{\frac{1}{\gamma}} > y_\tau^{a,SS}$*. The merger adds value:*

$$V^{a,m} = 2\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{\gamma+1}{\gamma}} [Z_A Z_B]^{\frac{1}{2\gamma}} > V_A^{a,SS} + V_B^{a,SS}.$$

Theorem 8 shows mergers add value to the innovative process. By merging, the joint firm chooses innovation intensities greater than those that the two entrepreneurs would choose individually. Because of the positive externality between investment levels, y_τ, inefficiently low levels of investment occur when each entrepreneur maximizes her own payoff. By merging, the post-acquisition firm internalizes the positive spillover effects of investment, leading to greater innovation and firm value. Theorem 8 shows synergies are endogenous, and generated by this valuation externality.

We now examine the impact of mergers at the interim date, $t = 2$, on the entrepreneurs' ex-ante incentives to innovate. The initial innovation decision depends on expectations of merger terms. The acquisition price depends on allocation of the surplus between the two entrepreneurs. Allocation of the synergies created in the merger occurs through bargaining, and we assume the two entrepreneurs will split the surplus equally.[30] Thus, if both innovations are successful in the first stage, entrepreneur τ earns $\Upsilon_\tau^{a,m} = \mathcal{U}_\tau^{a,SS} + \frac{1}{2}\left(\mathcal{U}^{a,m} - \mathcal{U}_A^{a,SS} - \mathcal{U}_B^{a,SS}\right)$.

Theorem 9 *There are thresholds* $\left\{\underline{k}_\tau^{a,m}, \bar{k}_\tau^{a,m}\right\}$ *(defined in the Appendix) with* $\underline{k}_\tau^{a,m} < \bar{k}_\tau^{a,m}$*, such that: (i) for low levels of discover cost,* $k_\tau \leq \underline{k}_\tau^{a,m}$*, an entrepreneur always innovates,* $d_\tau^a = 1$*; (ii) for high levels of discovery cost,* $k_\tau \geq \bar{k}_\tau^{a,m}$*, an entrepreneur never innovates,* $d_\tau^a = 0$*; (iii) for intermediate levels of the discovery cost,* $k_\tau \in \left(\underline{k}_\tau^{a,m}, \bar{k}_\tau^{a,m}\right)$*, an entrepreneur innovates only if the other entrepreneur innovates as well,* $d_\tau^a = d_{\tau'}^a$*. If both entrepreneurs have intermediate levels of discovery costs, there are two subgame perfect equilibria, one where both entrepreneurs innovate,* $d_A^a = d_B^a = 1$*, and one where neither innovate,* $d_A^a = d_B^a = 0$*. The innovation equilibrium Pareto-*

[30] It is easy to see that this surplus allocation will be the same as the one obtained through Shapley Values.

dominates the no-innovation equilibrium. Finally $\underline{k}_{\tau}^{a,m} = \underline{k}_{\tau}^{a} < \bar{k}_{\tau}^{a} < \bar{k}_{\tau}^{a,m}$*: the possibility of a merger induces entrepreneurs to innovate more ex ante.*

Theorems 8 and 9 show an active M&A market promotes innovative activity and leads to greater innovation rates, improved investor beliefs, and higher firm valuations. Synergies created are a direct consequence of endogenous investor beliefs due to uncertainty aversion. A merger allows entrepreneurs to internalize the positive impact that the choice of the innovation intensity in one innovation has on other innovations, and leads to greater innovation rates. Thus, the merger of innovations endogenously promotes stronger investor sentiment leading to greater valuations.[31]

6 Empirical Implications

We propose a parsimonious theory of innovation waves and investor sentiment based on uncertainty aversion. Our model leads to several novel empirical implications.

1. Innovation waves localized in specific (technological) sectors. Strategic complementarity between entrepreneurs' innovation decisions creates the possibility of innovation waves. Arrival of innovation opportunities (i.e. project-ideas) in the economy may depend on classic "fundamentals" such as random technological advances in certain sectors, such as Information Technologies or Life Sciences. Our paper suggests that such technological advances, while necessary, may not be sufficient to start a wave. Rather, an innovation wave occurs when a critical mass of potential innovators is attained which will spur a "hot" market for innovative companies.

Our model can be applied more broadly to spillovers across industries. Specifically, an innovation wave may start in one "sector" and then spill over to other unrelated "sectors."[32] This can happen, for example, when a positive shock to entrepreneurs in one sector lowers their discovery cost from a high level, $k_{\tau} > \bar{k}_{\tau}$, to a low level, $k_{\tau} < \underline{k}_{\tau}$, while the other entrepreneur faces a moderate discovery cost, $k_{\tau'} \in \left(\underline{k}_{\tau'}, \bar{k}_{\tau'}\right)$, $\tau \neq \tau'$. If the discovery costs of the first set of entrepreneurs decrease to a low level, $k_{\tau} < \underline{k}_{\tau}$, it now becomes optimal for them to initiate the innovation process. This decision makes it profitable for other entrepreneurs to innovate as well, in anticipation of higher

[31] Point 5 of Table 1 displays the case with acquisitions. The possibility of an acquisition raises the optimal innovation intensity, $y^{a,m}$, with respect to autonomy, $y^{a,SS}$, leading to greater equity valuations ($V_{\tau}^{a,m} > V_{\tau}^{a,SS}$).

[32] For example, a positive technological shock to LinkedIn may boost Uber, even if no direct link is present.

equity prices. Note that the spillover across sectors works through an "equity valuation" channel driven by more favorable investor beliefs, rather than a pure technological channel. Similar results hold for the productivity of innovation, Z_τ, and the probability of success, q_τ.

2. Investor sentiment, hot IPO markets, and equity returns. Cyclical "hot and cold" markets for IPOs have been documented in the literature, and they largely remain a puzzle (see, for example, Ritter and Welch, 2002). Lee, Shleifer, and Thaler (1991) document a greater volume of IPOs in periods of strong investor sentiment. Lowry (2003) finds "hot" IPO markets are associated with strong investor sentiment and high demand for capital by firms. Helwege and Liang (2004) document hot and cold IPOs are largely concentrated in the same narrow set of industries and reflect greater investor optimism. In our model, the market value of an entrepreneur's firm is increasing in the number of successful firms in the market and on their demand for external capital, because uncertainty-averse investors are more optimistic when they can invest in the equity of a larger set of new firms, leading to higher equity valuations. Innovation waves are associated with improved investor sentiment toward innovations and with booms in the equity of technology firms, which are then followed by lower stock returns. Thus, our model can explain the relationship between IPO volume, stock market valuations, and the subsequent lower returns documented in the literature.

Our paper can also help explain the observed negative relationship between capital investments and subsequent negative equity returns, especially for small firms (Lamont, 2000, Titman et al., 2004, and Cooper et al., 2008). In our model, the negative association between investment and stock returns is not driven by time-varying risk premia, but rather by changes of future expected cash flow due to a change in investor sentiment. Eberhart et al. (2005), Hirschleifer et al. (2013), in contrast, find a *positive* relation between firm-level innovation and future equity returns, and suggest investors are slow in recognizing the full benefits of innovation. Similar patterns are observed in Kumar and Li (2016), who suggest investments in innovative capacity create new investment opportunities (i.e., real options), increasing firm risk profile and, thus, discount rates. Our paper predicts the negative association between capital investments and subsequent equity returns should be stronger during innovation waves within an industry. This means the relation between investment in innovation and future equity returns should differ when on or off the wave. This is a novel and testable prediction.

3. Venture capitalists portfolios and innovation rates. An additional implication of our model is a new role for VCs and the formation of their portfolios. By investing in firms in the same industry, as opposed to diversifying in unrelated industries as prescribed by traditional risk aversion, VCs can increase the benefits of uncertainty hedging and offer relatively better terms to entrepreneurs.[33] Thus, entrepreneurs in more uncertain industries should seek financing from focused VCs (whereby the uncertainty-hedging effect dominates), while entrepreneurs in less uncertain industries should seek financing from generalist VCs (whereby the risk diversification effect dominates). In addition, high level of uncertainty, by promoting focused portfolio, promotes (and rewards) investment in sector-specific human capital by VCs. This is a new and testable implication.

Our model identifies an additional role for VCs: improving coordination across entrepreneurs. If discovery costs fall in the intermediate range, $k_\tau \in \left(\underline{k}_\tau, \bar{k}_\tau\right)$, entrepreneurs face an "assurance game:" each entrepreneur will be willing to incur the discovery cost and innovate only if she is assured other entrepreneurs will also do the same. Lacking such assurance, entrepreneurs may be confined to the inefficient equilibrium with no innovation. In this setting, a VC may play a positive role by addressing the coordination failure among entrepreneurs. By investing in several technology firms in the same industry, the VC can help coordination among entrepreneurs, leading to greater innovation rates.[34] The innovation cycle discussed in this paper can also generate a VC cycle: periods characterized by strong investor sentiment and a hot IPO market will also be associated with strong VC fundraising and commitment activity, as documented in Gompers et al. (2008).

4. Innovation and competition. Our model suggests competition has two effects on innovation activity in an economy. First, greater competition makes it less attractive for an entrepreneur to wait for a wave, thus accelerating the innovation process. Second, the presence of competing innovators has a beneficial effect of investors' beliefs (due to uncertainty hedging), with a positive effect on equity prices (all else equal). This implies the impact of competition on the incentive to innovate depends on both the extent of uncertainty and the strength of competition. For low levels of competition, innovation decisions are strategic complements and entrepreneurs benefit from waiting to innovate in order to join a wave. In contrast, for high levels of competition, innovation

[33] Zider (1998) argues that a primary strategy for VCs is to focus on promising industries, as opposed to just investing in promising firms. Kolarich (2019) suggests that investing in a small concentrated portfolio, a strategy known has "conviction investing," better positions a VC for large gains (i.e., finding a "unicorn"). The advantages and disadvantages for a VC investing in diversified portfolios are further discussed in Gery (2018).

[34] A positive impact of VC financing on innovation has been documented in Kortum and Lerner (2000) and Hellmann and Puri (2000) among others (see Da Rin, Hellmann and Puri, 2013, for a extensive survey).

decisions are strategic substitutes and entrepreneurs innovate immediately to limit the adverse impact of competition. The negative effect of competition on innovation waves is dampened by the presence of uncertainty: competition accelerates innovation in mature industries, characterized by relatively low uncertainty, but promotes innovation waves in new industries characterized by high uncertainty. These properties leads to the following empirical implications. First, we find an inverse U-Shaped relationship between competition and innovation rates, a feature consistent with existing empirical literature (e.g., Aghion et al., 2005). Second, and more interestingly, we show that the relation between competition and the size of innovation waves is a U-shaped function, especially in industries characterized by greater uncertainty. These are new and testable implications.

5. *Innovation, investor sentiment, and merger activity.* Our paper presents a new channel in which merger activity can generate synergies and spur innovation. Synergistic gains are the outcome of the beneficial spillover effect of a merger on the expected value of the innovation. In the post-merger firm, innovators choose greater innovation rates. Our model also predicts merger activity involving innovative firms will be associated with strong investor sentiment and greater valuations. These results are consistent with Bena and Li (2014): synergies obtained in combining innovation capabilities are important drivers of acquisitions, with an overall positive impact on innovation rates. A positive relation between merger waves and strong market valuations has been documented in several empirical studies, such as Shleifer and Vishny (2003), Rhodes–Kropf, Robinson and Viswanathan (2005), Dong et al. (2006), and Rosen (2006), among others.

6. Incubators. Our model also provides a new motivation for technological incubators. Incubators allow entrepreneurs to meet each other, and coordinate innovation decisions. Our model made the standard assumption of full information, but this may not hold in practice. In an incubator, entrepreneurs can meet each other, perhaps overcoming coordination failure. This implies incubators will be particularly valuable in industries surrounded by great uncertainty, a novel prediction.

7 Conclusion and Future Research

In this paper, we show uncertainty aversion generates endogenous investor beliefs (or sentiment), resulting in innovation waves. Because of uncertainty hedging, investors treat different uncertain investments as complements, creating strategic complementarity in entrepreneurial behavior that results in innovation waves. Our model can explain why there are some periods when investment in innovation is "hot," and investors are more willing to invest in risky investment projects tainted by significant uncertainty.

In our model, we make several simplifying assumptions to facilitate analysis. Explicit examination of models relaxing such assumptions is, however, beyond the scope of this paper. Thus, our paper could be extended in several important dimensions.

First, we develop our model under the simplifying assumptions that entrepreneurs are risk and uncertainty neutral. Entrepreneurial uncertainty and risk aversion can be introduced in the analysis at the cost of adding additional complexities which would not change the nature of our results. First, entrepreneurial uncertainty aversion will have only a negligible impact on our model because entrepreneurs are exposed uniquely to the uncertainty of their own firms. This means that entrepreneurs would use their "worst-case" scenario to assess the success probability of the innovation's first stage, leaving our results unchanged. In contrast, risk-averse entrepreneurs will be concerned with the risk they face in their innovations, and they will require again a risk premium to initiate the innovation process. Interestingly, risk aversion of entrepreneurs amplifies the effect of investor uncertainty aversion. Risk-averse entrepreneurs will also be concerned about the likelihood that other entrepreneurs are successful, since the presence of other entrepreneurs will affect the market value of their firms at the interim date, given investors' uncertainty aversion. This channel would complement the one we analyze in this paper, which is centered on investors.

Second, in our paper it is quite useful that the core belief set $\mathcal{M}$ is strictly convex with a smooth boundary. Although this assumption greatly simplifies the analysis, it is not critical for our results. The main results of our paper depend only on the benefits of uncertainty hedging, a feature at the very heart of uncertainty aversion. By holding ambiguous assets in a portfolio, uncertainty hedging offers an uncertainty-averse investor an advantage analogous to the benefits of risk diversification in standard portfolio theory. In our context, loosely speaking, the benefits of uncertainty hedging are lost either when the worst-case probability for each investment in problem (2) can be taken on a case-by-case basis, independently from investor's overall portfolio composition (which would happen, for example, in the case of "rectangular" core-beliefs sets), or when there is a single source of uncertainty affecting all assets. This extreme situation is analogous to the loss of the benefits of diversification when assets are perfectly positively correlated in traditional portfolio theory.

Note that in our paper, for generality, we take the core-beliefs set of investors as a representation of their primitive preferences. The core-beliefs set, however, could be obtained as the outcome of a "micro-foundation" that builds directly on investors' uncertainty on economic fundamentals. In Appendix B, we present a model specification that generates qualitatively identical results, where the source of uncertainty is consumer demand (formally, the proportion of consumers that exhibit a relatively stronger preference for each good). This specification generates sector-specific innovation waves as described in our paper. Detailed analysis of technological and economic drivers of uncertainty is an important and very promising avenue of future research.

Finally, an important feature we deliberately ignore are the effects of learning. Learning about either technologies or the economic environment is clearly a key component of the innovation process. Our model suggests that, due to the complementarity we identify, learning in one project (or sector) may have important spillover effects in other projects (or sectors). Also, learning may impact the extent of uncertainty present in the economy, affecting valuations, project investments, and investor predispostion toward innovation. We leave these considerations to future research.

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A Appendix: Proofs

Proof of Lemma 1. Let $x = \{x_A, x_B\}$ be indicator variables for success of type A and B assets: $x \in \{0,1\}^2$. If the probability of success is $p = \{p_A, p_B\}$ the probability of x is $p_A^{x_A} p_B^{x_B} (1-p_A)^{1-x_A} (1-p_B)^{1-x_B}$. Thus,

$$R(p|\hat{p}) = \sum_{x\in\{0,1\}^2} p_A^{x_A} p_B^{x_B} (1-p_A)^{1-x_A} (1-p_B)^{1-x_B} \ln \frac{p_A^{x_A} p_B^{x_B} (1-p_A)^{1-x_A} (1-p_B)^{1-x_B}}{\hat{p}_A^{x_A} \hat{p}_B^{x_B} (1-\hat{p}_A)^{1-x_A} (1-\hat{p}_B)^{1-x_B}}.$$

Because the log of a product is the sum of the logs, and probabilities sum to one, $R(p|\hat{p}) = R(p_A|\hat{p}_A) + R(p_B|\hat{p}_B)$, where $R(p_\tau|\hat{p}_\tau) = p_\tau \ln \frac{p_\tau}{\hat{p}_\tau} + (1-p_\tau)\ln\frac{1-p_\tau}{1-\hat{p}_\tau}$. Because $\frac{\partial^2 R}{\partial p_\tau^2} = \frac{1}{p_\tau} + \frac{1}{1-p_\tau}$, $R(p_\tau|\hat{p}_\tau)$ is strictly convex in p_τ. Thus, $R(p|\hat{p})$ is strictly convex in $p = \{p_A, p_B\}$. Also, $\lim_{p_\tau\to 0^+} R(p_\tau|\hat{p}_\tau) = \ln\frac{1}{1-\hat{p}_\tau}$ and $\lim_{p_\tau\to 1^-} R(p_\tau|\hat{p}_\tau) = \ln\frac{1}{\hat{p}_\tau}$. Define $\tilde{\eta}^0(\hat{p}) = \min_{\chi\in Q}\ln\frac{1}{\chi}$, where $Q = \{\hat{p}_A, 1-\hat{p}_A, \hat{p}_B, 1-\hat{p}_B\}$. Therefore, if $\tilde{\eta} < \tilde{\eta}^0(\hat{p})$, $\mathcal{M}$, as the lower level set of a strictly convex function, is strictly convex. Note this generalizes: Theorem 2.5.3 of Cover and Thomas (2006) shows relative entropy is additively separable in independent variables, and Theorem 2.7.2 shows it is strictly convex.

Suppose an investor receives y_A if $x_A = 1$ and y_B if $x_B = 1$, both strictly positive. R achieves a minimum of zero at $p = \hat{p}$. Because R is strictly convex in p, $\frac{\partial R}{\partial p_\tau} < 0$ for $p_\tau < \hat{p}_\tau$ and $\frac{\partial R}{\partial p_\tau} > 0$ for $p_\tau > \hat{p}_\tau$. The worst-case scenario solves $\min\{p_A y_A + p_B y_B\}$ subject to $R(p|\hat{p}) \le \tilde{\eta}$. Let λ be the multiplier and L be the Lagrangian: $L = -(p_A y_A + p_B y_B) - \lambda(R(p|\hat{p}) - \tilde{\eta})$, so $\frac{dL}{dp_\tau} = -y_\tau - \lambda\frac{\partial R}{\partial p_\tau}$. Because $y_\tau > 0$, $\frac{dL}{dp_\tau} = 0$ requires $\lambda\frac{\partial R}{\partial p_\tau} < 0$, $\lambda > 0$ and $\frac{\partial R}{\partial p_\tau} < 0$, so $p_\tau < \hat{p}_\tau$. If the investor has strictly positive exposure to only one innovation, $y_\tau > 0$ but $y_{\tau'} = 0$, the worst-case scenario solves $R(p_\tau|\hat{p}_\tau) = \tilde{\eta}$ for $p_\tau < \hat{p}_\tau$, and $p_{\tau'} = \hat{p}_{\tau'}$. If $y_A = y_B = 0$, claim holds WLOG. ■

Proof of Lemma 2. $U(\Pi) = \min u(\theta;\Pi)$ s.t. $\sum|\theta_\tau - \theta^*| \le \eta$. If $\omega_\tau y_\tau > 0 = \omega_{\tau'} y_{\tau'}$, $\frac{\partial u}{\partial\theta_\tau} > 0 = \frac{\partial u}{\partial\theta_{\tau'}}$, so $\theta_\tau = \theta^* - \eta$ and $\theta_{\tau'} = \theta^*$. If $\omega_\tau y_\tau, \omega_{\tau'} y_{\tau'} > 0$, let λ be the multiplier and L be the Lagrangian: $\frac{\partial L}{\partial\theta_\tau} = -e^{\theta_\tau - \theta_M}\omega_\tau y_\tau - \lambda sign(\theta_\tau - \theta^*)$. Because $\omega_\tau y_\tau > 0$ and $\lambda \ge 0$, $\theta_\tau \le \theta^*$. Because u is strictly convex in θ, FOCs are sufficient for a minimum. Because $\lambda > 0$, the constraint binds; substituting into $\frac{\partial L}{\partial\theta_\tau}|_{\theta_\tau = \check{\theta}_\tau} = 0$ yields (8). Thus, if $\check{\theta}^a_\tau(\Pi) \in [\theta^* - \eta, \theta^*]$, $\theta^a_\tau = \check{\theta}^a_\tau$. If $\check{\theta}^a_\tau < \theta^* - \eta$, $\frac{\partial L}{\partial\theta_\tau} < 0$ for $\theta_\tau \in [\theta^* - \eta, \theta^*]$, so $\theta^a_\tau = \theta^* - \eta$. If $\check{\theta}^a_\tau > \theta^*$, $\frac{\partial L}{\partial\theta_\tau} > 0$ for $\theta_\tau \in [\theta^* - \eta, \theta^*]$, so $\theta^a_\tau = \theta^*$. Therefore, (9) is the worst-case scenario. ■

Proof of Theorem 2. Each investor's objective is $U(\Pi) = \min_{\theta \in C} u(\theta; \Pi)$, where u is from (6). Thus, $\frac{dU}{d\omega_\tau} = \frac{\partial u}{\partial \omega_\tau} + \frac{\partial u}{\partial \theta_A}\frac{d\theta_A}{d\omega_\tau} + \frac{\partial u}{\partial \theta_B}\frac{d\theta_B}{d\omega_\tau}$. If investors are uncertainty-neutral, the last two terms disappear (θ_τ is constant). If uncertainty averse, $\overrightarrow{\theta}^a$ solves the minimization problem. For interior solutions, by Lemma 2, $\frac{\partial u}{\partial \theta_A} = \frac{\partial u}{\partial \theta_B} = \lambda$, so the last two terms sum to $\lambda \frac{\partial(\theta_A + \theta_B)}{\partial \omega_\tau} = 0$ because $\theta_A + \theta_B = 2\theta^* - \eta$ is constant. For corner solutions, $\frac{\partial \theta_A}{\partial \omega_\tau} = \frac{\partial \theta_B}{\partial \omega_\tau} = 0$. Therefore, $\frac{dU}{d\omega_\tau} = \frac{\partial u}{\partial \omega_\tau} = p(\theta_\tau^a) y_\tau - V_\tau$: market clearing implies $V_\tau = p(\theta_\tau^a) y_\tau$, and all investors have the same θ_τ^a. By Lemma 2, $\frac{\omega_A}{\omega_B}$ is constant across investors. Because $\frac{y_\tau}{y_{\tau'}} \in (e^{-\eta}, e^{\eta})$, prices follow by substitution. ■

Proof of Corollary 1. $y_\tau \in (e^{-\eta} y_{\tau'}, e^{\eta} y_\tau)$, so $V_\tau^{a,SS} = p(\theta^* - \frac{\eta}{2}) y_\tau^{\frac{1}{2}} y_{\tau'}^{\frac{1}{2}} > p(\theta^* - \eta) y_\tau = V_\tau^{a,SF}$: an entrepreneur sells innovation for more when the other innovation is available.[35] $\mathcal{U}_\tau = V_\tau - c_\tau$, so claim on utility follows. ■

Proof of Theorem 3. $E\mathcal{U}_\tau^{a,I} \geq 0$ iff $k_\tau \leq \bar{k}_\tau^a \equiv q_\tau q_{\tau'} \mathcal{U}_\tau^{a,SS} + q_\tau (1 - q_{\tau'}) \mathcal{U}_\tau^{a,SF}$, and $E\mathcal{U}_\tau^{a,N} \geq 0$ iff $k_\tau \leq \underline{k}_\tau^a \equiv q_\tau \mathcal{U}_\tau^{a,SF}$. $\mathcal{U}_\tau^{a,SS} > \mathcal{U}_\tau^{a,SF}$, so $\underline{k}_\tau^a < \bar{k}_\tau^a$. If $k_\tau \in [\underline{k}_\tau^a, \bar{k}_\tau^a]$ for both entrepreneurs, both equilibria exist. If $d_A^a = d_B^a = 0$, entrepreneurs earn zero. If $d_A^a = d_B^a = 1$, entrepreneurs earn $E\mathcal{U}_\tau^{a,I} \geq 0$ (strict inequality if $k_\tau < \bar{k}_\tau^a$). Firms are priced so investors are indifferent, so innovation equilibrium dominates the other. ■

Proof of Corollary 2. Comparative statics follow from inspection of $\underline{k}_\tau^a$ and $\bar{k}_\tau^a$, and because $\mathcal{U}_\tau^{a,SS}$ is increasing in y_τ and $y_{\tau'}$, $\mathcal{U}_\tau^{a,SF}$ is increasing in y_τ, and $\mathcal{U}_\tau^{a,SS} > \mathcal{U}_\tau^{a,SF}$. ■

Proof of Lemma 3. At $t-1$, entrepreneurs in S_t chose to implement their project-ideas and had a successful first-stage innovation. Only implemented projects can be traded, so investors choose portfolio weights $\{\omega_n\}_{n \in S_t}$ to maximize their minimum expected payoff, $\min_{\overrightarrow{\theta} \in C} u(\overrightarrow{\theta})$, where $u(\overrightarrow{\theta}) = \sum_{n \in S_t} \omega_n [\delta p_n(\theta_{nt}) y_n - V_n] + \omega_0$. By identical proof to Theorem 2, in equilibrium, $\omega_n = 1$ for all $n \in S_t$ and $V_n = \delta p_n(\theta_{nt}^a) y_n$. Recall that $\overrightarrow{\theta} \in C$ iff $\sum_{n \in S_t} |\theta_n - \theta^*| \leq \eta$. Let L be the Lagrangian function for the minimization problem and let λ be the multiplier for the sum. Thus, $\frac{\partial L}{\partial \theta_{nt}} = -e^{\theta_{nt} - \theta_M} y_n - \lambda sign(\theta_{nt} - \theta^*)$. For all $n \in S_t$, $y_n = y > 0$, so by complementary slackness, $\lambda \geq 0$, and thus $\theta_{nt} < \theta^*$. $\frac{\partial L}{\partial \theta_{nt}} = 0$ iff $e^{\theta_{nt} - \theta_M} y_n = \lambda$, so θ_{nt} is constant for all $n \in S_t$: $\theta_{nt} = \theta^* - \frac{\eta}{s_t}$, because $s_t = |S_t|$.[36] Market valuation is increasing in s_t because θ_{nt}^a is increasing in s_t. ■

Proof of Theorem 4. We consider Markov Perfect Equilibria with symmetric pure strategies. The state space is the number of entrepreneurs with projects, the action space is to develop or not, $d_n(\nu_t) \in \{0, 1\}$, the payoff solves (21). The transition probability is as follows: let $D_t = \sum_{n \in \mathcal{E}_t} d_n(\nu_t)$. Because D_t projects are developed at time t, and a new project arrives with probability π, $P(\nu_{t+1} = \nu_t - D_t + 1) = \pi$ and $P(\nu_{t+1} = \nu_t - D_t) = 1 - \pi$. The discount factor is $\delta \in (0, 1)$ and, for simplicity, we assume that the initial state is $\nu_0 = 0$.

If s entrepreneurs develop, they each earn $\hat{u}(s) = \delta^2 p(\theta^* - \frac{\eta}{s}) y - \delta c - k$, increasing in s. To have innovation in equilibrium, $\hat{u}(s) > 0$ for equilibrium s, so $\lim_{s \to \infty} \hat{u}(s) > 0$,[37] or equivalently, $k < \bar{k}_d \equiv \delta^2 p(\theta^*) y - \delta c$ (if $k > \bar{k}_d$, innovation is not profitable and never occurs). If each entrepreneur develops her innovation immediately upon discovering it, she earns utility $\hat{u}(1)$. Note $\hat{u}(1) \leq 0$ iff $k \geq k_0 \equiv \delta^2 p(\theta^* - \eta) y - \delta c$. Define $\nu_0 \equiv \min\{s | \hat{u}(s) > 0, s \in \mathbb{N}\}$.

Suppose an entrepreneur believes other entrepreneurs develop according to $d_n^*(\nu)$, for $\nu \in \mathbb{N}$. Because $d_n^* \in \{0, 1\}$, let $\nu^* \equiv \inf\{\nu \in \mathbb{N} | d_{n-}^*(\nu) = 1\}$. Entrepreneurs can forgo innovation, earning 0, so $\hat{u}(\nu^*) > 0$, so $\nu^* \geq \nu_0$. We solve (21) assuming the entrepreneur also plays d_n^*, then verify d_n^* is optimal. Because $\nu > \nu^*$ is off-equilibrium, $\mathcal{U}_n^a(\nu, d_{n-}^*)$ is undefined for $\nu > \nu^*$. Everyone develops when $\nu_t = \nu^*$, so $\mathcal{U}_n^a(\nu^*, d_{n-}^*) = \hat{u}(\nu^*)$. For $\nu_t < \nu^*$, $d_n^* = 0$, so $\mathcal{U}_n^a(\nu_t, d_{n-}^*) = \delta E\mathcal{U}_n^a(\nu_{t+1}, d_{n-}^*)$ and $E\mathcal{U}_n^a(\nu_{t+1}, d_{n-}^*) = \pi \mathcal{U}_n^a(\nu_t + 1, d_{n-}^*) + (1 - \pi) \mathcal{U}_n^a(\nu_t, d_{n-}^*)$, which implies $\mathcal{U}_n^a(\nu_t, d_{n-}^*) = \frac{\delta\pi}{1 - \delta(1 - \pi)} \mathcal{U}_n^a(\nu_t + 1, d_{n-}^*)$. Because this holds for all $\nu_t < \nu^*$, $\mathcal{U}_n^a(\nu_t, d_{n-}^*) = \left[\frac{\delta\pi}{1 - \delta(1 - \pi)}\right]^{\nu^* - \nu_t} \hat{u}(\nu^*)$.

We must show innovating is optimal when $\nu_t = \nu^*$, and waiting optimal when $\nu_t < \nu^*$. Because $\hat{u}(\nu^*) > 0$ and $\frac{\delta\pi}{1 - \delta(1 - \pi)} < 1$, $\mathcal{U}_n^a(\nu_t, d_{n-}^*)$ is positive and increasing in ν_t. When everyone else innovate, $\nu_t = \nu^*$, if an entrepreneur sets $d = 0$, her expected utility is $\mathcal{U}_n^a(1, d_{n-}^*) < \mathcal{U}_n^a(\nu^*, d_{n-}^*)$, so she innovates as well. Also, entrepreneurs must not

[35] Note the result follows weakly even without restrictions on $\frac{y_\tau}{y_{\tau'}}$: if $y_\tau \geq e^{\eta} y_{\tau'}$, $V_\tau^{a,SS} = V_\tau^{a,SF}$, while if $y_\tau < e^{-\eta} y_{\tau'}$, $V_\tau^{a,SS} = e^{\theta^* - \theta_M} y_\tau > V_\tau^{a,SF}$.

[36] By identical argument, undeveloped projects are not traded this period, so investors treat those projects as if $y_n = 0$. Thus, investors will assess any undeveloped or untraded project with $\theta_{nt} = \theta^*$.

[37] $\hat{u}(s)$ is increasing in s and bounded, $\lim_{s \to +\infty} \hat{u}(s) = \delta^2 e^{\theta^* - \theta_M} y - \delta c - k < +\infty$, so the entrepreneur will not hold a project indefinitely in hopes of earning an infinite payoff.

prefer to innovate immediately. Because $\mathcal{U}_n^a$ is increasing in ν_t, it is sufficient that the first entrepreneur prefers to wait: $\hat{u}(1) \leq \mathcal{U}_n^a\left(1, d_{n-}^*\right)$. Define $\Upsilon(\nu^*) = \mathcal{U}_n^a\left(1, d_{n-}^*\right)$: $\frac{\Upsilon'}{\Upsilon} = \ln\left(\frac{\delta\pi}{1-\delta(1-\pi)}\right) + \frac{\delta V_{nt}^a(\nu^*)}{\delta V_{nt}^a(\nu^*)-\delta c-k}\frac{\eta}{(\nu^*)^2}$. Because $\hat{u}(\nu^*) > 0$, the second term is the product of two decreasing functions; $\ln\left(\frac{\delta\pi}{1-\delta(1-\pi)}\right) < 0$ does not depend on ν^*. Thus, if $\Upsilon > 0$, $\Upsilon' > 0$ for $\nu < \hat{\nu}$ and $\Upsilon' < 0$ for $\nu > \hat{\nu}$, for some $\hat{\nu}$. Because $\lim_{s\to\infty}\hat{u}(s)$ is positive and finite, $\lim_{\nu^*\to\infty}\Upsilon(\nu^*) = 0$. Therefore, if $\hat{u}(1) \leq 0$, the set of equilibria are all waves $\nu^* \geq \underline{\nu} \equiv \min\{s|\hat{u}(s) > 0, s \in \mathbb{N}\}$.[38] In contrast, if $\hat{u}(1) > 0$, equilibria waves are $\nu^* \in [\underline{\nu}, \bar{\nu}]$, $\underline{\nu} \equiv \min\{s|\Upsilon(s) > \hat{u}(1), s \in \mathbb{N}\}$ and $\bar{\nu} = \max\{s|\Upsilon(s) > \hat{u}(1), s \in \mathbb{N}\}$.

Finally, consider the no wave equilibrium. An entrepreneur could unilaterally create a wave of two by waiting for the next entrepreneur. Thus, it is an equilibrium to innovate immediately only it $\hat{u}(1) > \Upsilon(2)$, or equivalently, if $k < \underline{k}_d \equiv \frac{1-\delta(1-\pi)}{1-\delta}\delta^2 p(\theta^* - \eta) y - \frac{\delta\pi}{1-\delta}\delta^2 p\left(\theta^* - \frac{\eta}{2}\right) y - \delta c$. Because $\hat{u}(1) = \Upsilon(1)$, $k < \underline{k}_d$ also implies that $\hat{\nu} < 2$, so that $\hat{u}(1) > \Upsilon(\nu)$ for all $\nu \geq 2$: if it is an equilibrium to innovate unilaterally, there are no wave equilibria. ■

Proof of Theorem 5. Claims on $\underline{\nu}$ and $\bar{\nu}$ follow because Υ is hump-shaped on relevant domain (proof of Theorem 4). An equilibrium is efficient (in a class) if it grants higher payoff to entrepreneurs than other equilibria (in that class), because investors are indifferent. Theorem 4 showed any wave equilibrium is a threshold equilibrium: entrepreneurs wait until there are ν^* projects to innovate. The efficient equilibrium maximizes Υ. Define $\Delta\Upsilon = \Upsilon(\nu+1) - \Upsilon(\nu)$. Theorem 4 showed $\frac{\Upsilon'}{\Upsilon}$ is decreasing for positive Υ, so $\Upsilon' > 0$ for $\nu < \hat{\nu}$ and $\Upsilon' < 0$ for $\nu > \hat{\nu}$. For $\nu < \hat{\nu} - 1$, $\Delta\Upsilon > 0$, for $\nu > \hat{\nu}$, $\Delta\Upsilon < 0$. Therefore, there exists a ν^e such that for all ν with $\Upsilon(\nu) > 0$, if $\nu < \nu^e$, $\Delta\Upsilon > 0$ and if $\nu \geq \nu^e$, $\Delta\Upsilon \leq 0$. Note that $k < \bar{k}_d$ implies there exists ν such that $\Upsilon > 0$. ■

Proof of Corollary 3. $\underline{k}_d$ solves $f = \Upsilon(2) - \Upsilon(1) = 0$. Because $\frac{\partial f}{\partial k} = \frac{1-\delta}{1-\delta(1-\pi)} > 0$ and $\frac{\partial f}{\partial \pi} = \frac{\delta(1-\delta)}{(1-\delta+\delta\pi)^2}\hat{u}(2) > 0$, $\frac{d\underline{k}_d}{d\pi} < 0$. Also, $\frac{\partial f}{\partial \eta} = \delta^2 p(\theta^* - \eta) y\left(1 - \frac{1}{2}\frac{\delta\pi}{1-\delta(1-\pi)}e^{\frac{\eta}{2}}\right)$. If $\eta > 2\ln\frac{2[1-\delta(1-\pi)]}{\delta\pi}$, then $f > 0$ for all $k > 0$, so if there exists a positive $\underline{k}_d$ such that $f = 0$, $\frac{\partial f}{\partial \eta} > 0$, so $\frac{d\underline{k}_d}{d\eta} < 0$.

If $\hat{u}(1) \leq 0$, $\underline{\nu} \equiv \min\{s|\hat{u}(s) > 0, s \in \mathbb{N}\}$. Because $\hat{u}' > 0$, anything that increases $\hat{u}$ decreases $\underline{\nu}$, and vice versa. $\frac{\partial \hat{u}}{\partial \eta} = -\frac{1}{s}\delta^2 e^{\theta^* - \frac{\eta}{\underline{\nu}} - \theta_M} y < 0$ and $\frac{\partial \hat{u}}{\partial k} = -1 < 0$ so $\underline{\nu}$ is increasing in $\{\eta, k\}$. $\frac{\partial \hat{u}}{\partial \delta} = \delta e^{\theta^* - \frac{\eta}{\underline{\nu}} - \theta_M} y + \hat{u}(\underline{\nu}) + \frac{k}{\delta} > 0$, so $\underline{\nu}$ is decreasing in δ. If there are waves in equilibrium, yet $\hat{u}(1) > 0$, $\underline{\nu} = 2$, so claims trivially hold.

The optimal wave satisfies $\Delta\Upsilon > 0$ for $\nu < \nu^e$ and $\Delta\Upsilon \leq 0$ for $\nu \geq \nu^e$, where $\Upsilon(\nu) = \left(\frac{\delta\pi}{1-\delta(1-\pi)}\right)^{\nu-1}\hat{u}(\nu)$. $\frac{\partial \Delta\Upsilon}{\partial \eta} = -\frac{1}{\nu+1}\Delta\Upsilon + \frac{1}{\nu(\nu+1)}\Upsilon(\nu) + \frac{(\delta c+k)}{\nu+1}\left(\frac{\delta\pi}{1-\delta(1-\pi)}\right)^{\nu}\frac{1-\delta}{1-\delta(1-\pi)}$: the last two terms are positive, so if $\Delta\Upsilon|_\eta > 0$, then $\Delta\Upsilon|_{\hat{\eta}} > 0$ for all $\hat{\eta} > \eta$, so ν^e is increasing in η. $\frac{\partial \Delta\Upsilon}{\partial \pi} = \left(\frac{\delta\pi}{1-\delta(1-\pi)}\right)^{\nu-1}\frac{\delta(1-\delta)}{[1-\delta(1-\pi)]^2}\hat{u}(\nu+1) + (\nu-1)\frac{1-\delta}{[1-\delta(1-\pi)]\pi}\Delta\Upsilon(\nu)$: $\frac{\partial \Delta\Upsilon}{\partial \pi} > 0$ for $\nu \leq \nu^e$, so ν^e is increasing in π. $\frac{\partial \Delta\Upsilon}{\partial k} = \left(\frac{\delta\pi}{1-\delta(1-\pi)}\right)^{\nu-1}\frac{1-\delta}{1-\delta(1-\pi)} > 0$, so ν^e is increasing in k. ■

Proof of Theorem 6. Proof follows by identical reasoning to Theorem 4: state space, action space, discount factor, and initial state are identical. Different is that the transition probability and objective are affected by competition. Again, $D_t = \sum_{n\in\mathcal{E}_t} d_n(\nu_t)$. If $D_t < \nu_t$, $P(\nu_{t+1} = \nu_t - D_t + 1) = \pi$, $P(\nu_{t+1} = \nu_t - D_t) = 1 - \pi - \varepsilon$, and $P(\nu_{t+1} = 0) = \varepsilon$, while if $D_t = \nu_t$, $P(\nu_{t+1} = 1) = \pi$ and $P(\nu_{t+1} = 0) = 1 - \pi$. Entrepreneurs maximize (21), except in (20), $V_{nt}^{ac}(s) = \delta p\left(\theta^* - \frac{\eta}{s}\right)(1-\xi)^{s-1} y$. Because $\varepsilon > 0$, $E\mathcal{U}_n^a\left(\nu_{t+1}, d_{n-}^*\right) = \pi\mathcal{U}_n^a\left(\nu_t + 1, d_{n-}^*\right) + (1-\pi-\varepsilon)\mathcal{U}_n^a\left(\nu_t, d_{n-}^*\right)$, so $\mathcal{U}_n^a\left(\nu_t, d_{n-}^*\right) = \left[\frac{\delta\pi}{1-\delta(1-\pi-\varepsilon)}\right]^{\nu^*-\nu_t}\hat{u}(\nu^*)$. It is optimal to innovate unilaterally iff $\hat{u}(1) > \Upsilon(2)$ iff $k < \underline{k}_d^c \equiv \frac{1-\delta(1-\pi-\varepsilon)}{1-\delta(1-\varepsilon)}\delta^2 p(\theta^* - \eta) y - \frac{\delta\pi}{1-\delta(1-\varepsilon)}\delta^2 p\left(\theta^* - \frac{\eta}{2}\right)(1-\xi) y - \delta c$ iff $\xi > \underline{\xi} \equiv 1 - e^{-\frac{\eta}{2}}\frac{1-\delta(1-\pi-\varepsilon)}{\delta\pi} + \frac{1-\delta(1-\varepsilon)}{\delta\pi}\frac{\delta c+k}{\delta^2 p\left(\theta^* - \frac{\eta}{2}\right) y}$, so $\underline{k}_d^c(\xi) = \bar{k}_d^c(\xi)$ for $\xi > \underline{\xi}$. $\hat{u}'(s) = \delta^2 p\left(\theta^* - \frac{\eta}{s}\right)(1-\xi)^{s-1} y\left[\frac{\eta}{s^2} + \ln(1-\xi)\right] > 0$ iff $s < s^* \equiv \sqrt{\frac{\eta}{-\ln(1-\xi)}}$. Innovation is possible in equilibrium iff $k < \bar{k}_d \equiv \delta^2\max_{s\in\mathbb{N}}\left\{p\left(\theta^* - \frac{\eta}{s}\right)\xi^{s-1}\right\} y - \delta c$. Finally, $\frac{\Upsilon'}{\Upsilon} = \ln\left(\frac{\delta\pi}{1-\delta(1-\pi-\varepsilon)}\right) + \frac{\delta V_{nt}^{ac}(\nu^*)}{\delta V_{nt}^{ac}(\nu^*)-\delta c-k}\left(\frac{\eta}{(\nu^*)^2} + \ln(1-\xi)\right)$: $\frac{\Upsilon'}{\Upsilon}$ is decreasing in ν for all ν such that $\Upsilon > 0$. Thus, there exists ν^e such that $\Delta\Upsilon > 0$ for $\nu < \nu^e$ and $\Delta\Upsilon \leq 0$ for $\nu > \nu^e$. $\frac{\partial \underline{k}_d^c}{\partial \xi} = \frac{\delta\pi}{1-\delta(1-\varepsilon)}\delta^2 p\left(\theta^* - \frac{\eta}{2}\right) y > 0$ and $\frac{\partial^2 \underline{k}_d^c}{\partial\xi\partial\eta} = -\frac{1}{2}\frac{\delta\pi}{1-\delta(1-\varepsilon)}\delta^2 p\left(\theta^* - \frac{\eta}{2}\right) y < 0$. Let $f = \Upsilon(2) - u(1)$: $f = 0$ when $k = \underline{k}_d^c$ or $\xi = \underline{\xi}$. $\frac{\partial f}{\partial k} = \frac{1-\delta(1-\varepsilon)}{1-\delta(1-\pi-\varepsilon)}$, $\frac{\partial f}{\partial \xi} = -\frac{\delta\pi}{1-\delta(1-\pi-\varepsilon)}\delta^2 p\left(\theta^* - \frac{\eta}{2}\right) y$, and $\frac{\partial f}{\partial \varepsilon} = -\frac{\delta^2\pi}{[1-\delta(1-\pi-\varepsilon)]^2}\delta^2 p\left(\theta^* - \frac{\eta}{2}\right)(1-\xi) y$, so $\frac{d\underline{k}_d^c}{d\varepsilon} > 0$ and $\frac{d\underline{\xi}}{d\varepsilon} < 0$. ■

Proof of Corollary 4. $\frac{\partial \Delta\Upsilon}{\partial \xi} = \left(\frac{\delta\pi}{1-\delta(1-\pi-\varepsilon)}\right)^{\nu}\delta^2 p\left(\theta^* - \frac{\eta}{\nu+1}\right)\nu(1-\xi)^{\nu-2} y\left[\frac{\nu-1}{\nu}e^{-\frac{\eta}{\nu(\nu+1)}}\frac{1-\delta(1-\pi-\varepsilon)}{\delta\pi} - (1-\xi)\right]$,

[38] For $k \in \left(k_0, \bar{k}_d\right)$, never innovating is also an equilibrium, $\nu^* = +\infty$, because innovating unilaterally is unprofitable.

so $\frac{\partial \Delta\Upsilon}{\partial\xi} > 0$ iff $\xi > 1 - \frac{\nu-1}{\nu} e^{-\frac{\eta}{\nu(\nu+1)}} \frac{1-\delta(1-\pi-\varepsilon)}{\delta\pi}$, a cutoff increasing in ν. For $\nu = 1$, $\Delta\Upsilon$ is decreasing in ξ, because $\xi \in (0,1)$. For $\nu > 1$, $\Delta\Upsilon$ is U-Shaped in ξ, which implies $\{\xi|\Delta\Upsilon \leq 0\}$ is convex. Because $\left\{\xi|\nu \leq \nu^{opt}\right\} = \left\{\xi|\Delta\Upsilon\left(\nu^{opt}\right) \leq 0\right\}$, ν^{opt} is U-Shaped in ξ.

We first find the expected time between waves, then show that R is decreasing in wave size. Define the expected time until a wave when there are currently ν entrepreneurs as $T(\nu)$. Waves occur with ν^* entrepreneurs, so $T(\nu^*) = 0$. For $\nu < \nu^*$, $T(\nu_t) = E\left[T(\nu_{t+1}) + 1\right]$. Because a new entrepreneur arrives with probability π, but all projects become obsolete with probability ε, $ET(\nu_{t+1}) = \pi T(\nu_t + 1) + (1-\pi-\varepsilon) T(\nu_t) + \varepsilon T(0)$, which implies $T(\nu_t) = \frac{1}{\pi+\varepsilon} + \frac{\pi}{\pi+\varepsilon} T(\nu_t+1) + \frac{\varepsilon}{\pi+\varepsilon} T(0)$. It follows by induction that $T(\nu^* - k - 1) = \left[\frac{1}{\pi+\varepsilon} + \frac{\varepsilon}{\pi+\varepsilon} T(0)\right] \sum_{n=0}^{k} \left(\frac{\pi}{\pi+\varepsilon}\right)^n$: induction base is $\nu^* - 1$, because $T(\nu^*-1) = \frac{1}{\pi+\varepsilon} + \frac{\varepsilon}{\pi+\varepsilon} T(0)$, induction step follows by rearranging recursive definition. Rearranging, $T(0) = \frac{1}{\varepsilon}\left[\left(\frac{\pi+\varepsilon}{\pi}\right)^{\nu^*} - 1\right]$ and $T(\nu_t) = \frac{1}{\varepsilon}\left[\left(\frac{\pi+\varepsilon}{\pi}\right)^{\nu^*} - \left(\frac{\pi+\varepsilon}{\pi}\right)^{\nu_t}\right]$. Finally, because everyone innovates in a wave, $T(0)$ is the expected time between waves. Note $T(0)|_{\varepsilon=0} = \frac{\nu}{\pi}$, define $T^e = T(0)|_{\nu=\nu^e}$.

Note $\frac{\partial R^e}{\partial \nu^e} = \frac{\left(\frac{\pi+\varepsilon}{\pi}\right)^{\nu^e} - 1 - \nu^e \left(\frac{\pi+\varepsilon}{\pi}\right)^{\nu^e} \ln\left(\frac{\pi+\varepsilon}{\pi}\right)}{\frac{1}{\varepsilon}\left[\left(\frac{\pi+\varepsilon}{\pi}\right)^{\nu^e} - 1\right]^2}$, which is strictly negative because $\left(\frac{\pi+\varepsilon}{\pi}\right)^{\nu^e}$ is convex in ν^e. ξ affects R^e only through ν^e, so $\frac{dR}{d\xi} = \frac{dR}{d\nu^e}\frac{d\nu^e}{d\xi}$. Because $\frac{dR^e}{d\nu^e} < 0$, this implies R is inverse U-Shaped in ξ. Note $T(0)$ is the quotient differential of $\left(\frac{\pi+x}{\pi}\right)^{\nu^e}$, a convex function in x, so $T(0)$ is increasing in ε, and thus R^e is decreasing in ε. Finally, $\frac{\partial \Delta\Upsilon}{\partial \varepsilon} = -\nu \frac{\delta}{1-\delta(1-\pi)} \Delta\Upsilon - \frac{\delta}{1-\delta(1-\pi)} \Upsilon(\nu)$: $\frac{\partial \Delta\Upsilon}{\partial\varepsilon} < 0$ for $\nu \geq \nu^e$, so ν^e is decreasing in ε. ■

Proof of Theorem 7. When both entrepreneurs have successful first-stage,[39] they select y_τ to maximize $\mathcal{U}_\tau^{a,SS} = V_\tau^a - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$, V_τ^a from Theorem 2. For $y_\tau < e^{-\eta} y_{\tau'}$, $\frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} = p(\theta^*) - \frac{1}{Z_\tau} y_\tau^\gamma$. For $y_\tau \in \left(e^{-\eta} y_{\tau'}, e^{\eta} y_{\tau'}\right)$, $\frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} = \frac{1}{2} p\left(\theta^* - \frac{\eta}{2}\right) y_\tau^{-\frac{1}{2}} y_{\tau'}^{\frac{1}{2}} - \frac{1}{Z_\tau} y_\tau^\gamma$. For $y_\tau > e^{\eta} y_{\tau'}$, $\frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} = p(\theta^* - \eta) - \frac{1}{Z_\tau} y_\tau^\gamma$. Thus, $\lim_{y_\tau \uparrow e^{-\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} = p(\theta^*) - \frac{1}{Z_\tau}\left[e^{-\eta} y_{\tau'}\right]^\gamma > \frac{1}{2} p(\theta^*) - \frac{1}{Z_\tau}\left[e^{-\eta} y_{\tau'}\right]^\gamma = \lim_{y_\tau \downarrow e^{-\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau}$, but $\lim_{y_\tau \uparrow e^{\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} = \frac{1}{2} p(\theta^* - \eta) - \frac{1}{Z_\tau}\left[e^{\eta} y_{\tau'}\right]^\gamma < p(\theta^* - \eta) - \frac{1}{Z_\tau}\left[e^{\eta} y_{\tau'}\right]^\gamma = \lim_{y_\tau \downarrow e^{\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau}$. Thus, any critical point $y_\tau \leq e^{-\eta} y_{\tau'}$ is a global maximum, but a critical point in $\left(e^{-\eta} y_{\tau'}, e^{\eta} y_{\tau'}\right)$ must be compared to the critical point $y_\tau \geq e^{\eta} y_{\tau'}$.[40]

We now solve for the best-response function. It is optimal to select $y_\tau < e^{-\eta} y_{\tau'}$ only if $y_\tau = \left[p(\theta^*) Z_\tau\right]^{\frac{1}{\gamma}} < e^{-\eta} y_{\tau'}$. Thus, for $y_{\tau'} > \hat{y}_{\tau'} \equiv e^{\eta}\left[p(\theta^*) Z_\tau\right]^{\frac{1}{\gamma}}$, $y_\tau = \left[p(\theta^*) Z_\tau\right]^{\frac{1}{\gamma}}$. It is optimal to select $y_\tau = e^{-\eta} y_{\tau'}$ only if $\lim_{y_\tau \uparrow e^{-\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} \geq 0 \geq \lim_{y_\tau \downarrow e^{-\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau}$, which holds if $y_{\tau'} \in \left[\bar{y}_{\tau'}, \hat{y}_{\tau'}\right]$, $\bar{y}_{\tau'} \equiv \frac{1}{2^\gamma} \hat{y}_{\tau'}$. The optimal $y_\tau > e^{\eta} y_{\tau'}$ is $y_\tau^{a,SF}$, so $\mathcal{U}_\tau^{a,SF} = \left[p(\theta^* - \eta)\right]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma}$. For $y_\tau \in \left(e^{-\eta} y_{\tau'}, e^{\eta} y_{\tau'}\right)$ to be optimal, it must not only be locally optimal, but also must provide greater utility than $\mathcal{U}_\tau^{a,SF}$.[41] y_τ is a critical point, $\frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} = 0$, iff $y_\tau = \left[\frac{Z_\tau}{2} p\left(\theta^* - \frac{\eta}{2}\right) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma + \frac{1}{2}}}$, so $\mathcal{U}_\tau^{a,SS} = \left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{2\gamma+2}{2\gamma+1}} Z_\tau^{\frac{1}{2\gamma+1}} \left[\frac{1}{2}\right]^{\frac{1}{2\gamma+1}} \left[\frac{2\gamma+1}{2\gamma+2}\right] y_{\tau'}^{\frac{\gamma+1}{2\gamma+1}}$. $\mathcal{U}_\tau^{a,SS} > \mathcal{U}_\tau^{a,SF}$ iff $y_{\tau'} > \underline{y}_{\tau'} \equiv e^{-\eta\frac{2\gamma+1}{\gamma}} \frac{1}{2}\left[\frac{2\gamma}{2\gamma+1}\right]^{\frac{2\gamma+1}{\gamma+1}} \bar{y}_{\tau'}$. Thus, if $y_{\tau'} \leq \underline{y}_{\tau'}$, $y_\tau = y_\tau^{a,SF}$, while if $y_{\tau'} \in \left(\underline{y}_{\tau'}, \bar{y}_{\tau'}\right)$, $y_\tau = Y_\tau^{a,SS}(y_{\tau'})$.

Restricting attention to pure strategy equilibria, either $\frac{y_{\tau'}}{y_\tau} \in \left(e^{-\eta}, e^{\eta}\right)$ or not. Both entrepreneurs select innovation optimally. Because there is a kink at $y_{\tau'} = e^{\eta} y_\tau$, it can never be that $\frac{y_{\tau'}}{y_\tau} = e^{\eta}$ in equilibrium. Suppose to the contrary that so one entrepreneur selects $y_{\tau'} > e^{\eta} y_\tau$. In that case, $y_{\tau'} = \left[p(\theta^* - \eta)\right]^{\frac{1}{\gamma}} Z_{\tau'}^{\frac{1}{\gamma}}$, $y_\tau = \left[p(\theta^*)\right]^{\frac{1}{\gamma}} Z_\tau^{\frac{1}{\gamma}}$, so $y_{\tau'} > e^{\eta} y_\tau$ only if $\frac{Z_{\tau'}}{Z_\tau} > e^{\eta(\gamma+1)} > \psi = \frac{1}{4} e^{\eta(\gamma+1)}\left(1 + \frac{1}{2\gamma}\right)^{2\gamma}$ because $\left(1 + \frac{1}{2\gamma}\right)^{2\gamma} \in (1, e)$ for all $\gamma > 0$. Alternatively, if $\frac{y_{\tau'}}{y_\tau} \in \left(e^{-\eta}, e^{\eta}\right)$ in equilibrium, $Y_\tau^{a,SS}(y_{\tau'}) = \left[\frac{Z_\tau}{2} p\left(\theta^* - \frac{\eta}{2}\right) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$ for both

[39] If only entrepreneur τ has a successful first-stage, $y_{\tau'} = 0$. By Theorem 2, $V_\tau = p(\theta^* - \eta) y_\tau$, so the entrepreneur's payoff is $\mathcal{U}_\tau^{a,SF} = p(\theta^* - \eta) y_\tau - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$. $\frac{\partial \mathcal{U}_\tau^{a,SF}}{\partial y_\tau} = p(\theta^* - \eta) - \frac{1}{Z_\tau} y_\tau^\gamma$, and $\frac{\partial^2 \mathcal{U}_\tau^{a,SF}}{\partial y_\tau^2} = -\frac{\gamma}{Z_\tau} y_\tau^{\gamma-1} < 0$, so FOCs are sufficient. Thus, $y_\tau^{a,SF} = \left[p(\theta^* - \eta) Z_\tau\right]^{\frac{1}{\gamma}}$, $V_\tau^{a,SF} = \left[p(\theta^* - \eta)\right]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}}$, and $\mathcal{U}_\tau^{a,SF} = \left[p(\theta^* - \eta)\right]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma}$.

[40] If $\frac{y_\tau}{y_{\tau'}} \in \left(e^{-\eta}, e^{\eta}\right)$, $\frac{\partial^2 \mathcal{U}_\tau^{a,SS}}{\partial y_\tau^2} = -\frac{1}{4} p\left(\theta^* - \frac{\eta}{2}\right) y_\tau^{-\frac{3}{2}} y_{\tau'}^{\frac{1}{2}} - \frac{\gamma}{Z_\tau} y_\tau^\gamma$; if not, $\frac{\partial^2 \mathcal{U}_\tau^{a,SS}}{\partial y_\tau^2} = -\frac{\gamma}{Z_\tau} y_\tau^{\gamma-1}$. $\lim_{y_\tau \uparrow e^{-\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} > \lim_{y_\tau \downarrow e^{-\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau}$ but $\lim_{y_\tau \uparrow e^{\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} < \lim_{y_\tau \downarrow e^{\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau}$, so $\mathcal{U}_\tau^{a,SS}$ is concave everywhere except at $y_\tau = e^{\eta} y_{\tau'}$. Because $\lim_{y_\tau \uparrow e^{-\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau} > \lim_{y_\tau \downarrow e^{\eta} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^{a,SS}}{\partial y_\tau}$, any critical point $y_\tau \leq e^{-\eta} y_{\tau'}$ is a global maximum.

[41] This assumes that $y_\tau^{a,SF} > e^{\eta} y_{\tau'}$. If not, selecting $y_\tau^{a,SF}$ would give entrepreneur τ a strictly larger payoff than $\mathcal{U}_\tau^{a,SF}$, but a smaller payoff than $Y_\tau^{a,SS}(y_{\tau'})$, because $\mathcal{U}_\tau^{a,SS}$ is strictly concave on $\left(e^{-\eta} y_{\tau'}, e^{\eta} y_{\tau'}\right)$.

entrepreneurs, which implies $y_\tau^{a,SS} = \left[\frac{1}{2}p\left(\theta^* - \frac{\eta}{2}\right) Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$. This is better for entrepreneur τ than $y_\tau^{a,SF}$ only if $y_{\tau'}^{a,SS} > e^{-\frac{\eta}{2}\frac{2\gamma+1}{\gamma}}\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{1}{\gamma}} Z_\tau^{\frac{1}{\gamma}} 2^{\frac{1}{\gamma+1}}\left[\frac{2\gamma}{2\gamma+1}\right]^{\frac{2\gamma+1}{\gamma+1}}$, which holds because $\frac{Z_{\tau'}}{Z_\tau} > e^{-\eta(\gamma+1)}4\left[\frac{2\gamma}{2\gamma+1}\right]^{2\gamma} = \frac{1}{\psi}$. Therefore, when $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$, both entrepreneurs select innovation intensity with $\frac{y_{\tau'}}{y_\tau} \in \left(e^{-\eta}, e^{\eta}\right)$,[42] selecting $Y_\tau^{a,SS}\left(y_{\tau'}\right) = \left[\frac{Z_\tau}{2}p\left(\theta^* - \frac{\eta}{2}\right) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$, leading to equilibrium innovation $y_\tau^{a,SS} = \left[\frac{1}{2}p\left(\theta^* - \frac{\eta}{2}\right) Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$. Thus, the market price is $V_\tau^{a,SS} = 2^{-\frac{1}{\gamma}}\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{\gamma+1}{\gamma}}\left[Z_\tau Z_{\tau'}\right]^{\frac{1}{2\gamma}}$. Similarly, entrepreneur τ earns continuation utility $\mathcal{U}_\tau^{a,SS} = 2^{-\frac{1}{\gamma}}\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{\gamma+1}{\gamma}} Z_\tau^{\frac{1}{2\gamma}} Z_{\tau'}^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}$, for $\tau \in \{A, B\}$ and $\tau' \neq \tau$. ■

Proof of Corollary 5. Theorem 7 showed $\mathcal{U}_\tau^{a,SS} > \mathcal{U}_\tau^{a,SF}$ because $\frac{Z_{\tau'}}{Z_\tau} \in \left(\frac{1}{\psi}, \psi\right)$, where $\psi = \left[\frac{1}{4}e^{\eta}\right]^{\gamma+1}$. $V_\tau^{a,SS} = 2^{-\frac{1}{\gamma}}\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{\gamma+1}{\gamma}}\left[Z_\tau Z_{\tau'}\right]^{\frac{1}{2\gamma}}$ and $V_\tau^{a,SF} = \left[p\left(\theta^* - \eta\right)\right]^{\frac{\gamma+1}{\gamma}} Z_\tau^{\frac{1}{\gamma}}$, so $V_\tau^{a,SS} > V_\tau^{a,SF}$ iff $\frac{Z_{\tau'}}{Z_\tau} > 4e^{-\eta(\gamma+1)}$, which holds because $\frac{Z_{\tau'}}{Z_\tau} > \frac{1}{\psi} = 4^{\gamma+1}e^{-\eta(\gamma+1)}$. Similarly, $y_\tau^{a,SS} = \left[\frac{1}{2}p\left(\theta^* - \frac{\eta}{2}\right) Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$ and $y_\tau^{a,SF} = \left[p\left(\theta^* - \eta\right) Z_\tau\right]^{\frac{1}{\gamma}}$, so $y_\tau^{a,SS} > y_\tau^{a,SF}$ iff $\frac{Z_{\tau'}}{Z_\tau} > 4^{\gamma+1}e^{-\eta(1+\gamma)} = \frac{1}{\psi}$. Therefore, $\mathcal{U}_\tau^{a,SS} > \mathcal{U}_\tau^{a,SF}$, $V_\tau^{a,SS} > V_\tau^{a,SF}$, and $y_\tau^{a,SS} > y_\tau^{a,SF}$. ■

Proof of Theorem 8. The merged firm maximizes the combined value of the two projects. By Theorem 2, $V_A = V_B = p\left(\theta^* - \frac{\eta}{2}\right) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}}$, so the merged firm maximizes

$$\mathcal{U}^{a,m} = 2p\left(\theta^* - \frac{\eta}{2}\right) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}} - \frac{1}{Z_A\left(1+\gamma\right)} y_A^{1+\gamma} - \frac{1}{Z_B\left(1+\gamma\right)} y_B^{1+\gamma}.$$

Because $\frac{\partial \mathcal{U}^{a,m}}{\partial y_\tau} = p\left(\theta^* - \frac{\eta}{2}\right) y_\tau^{-\frac{1}{2}} y_{\tau'}^{\frac{1}{2}} - \frac{1}{Z_\tau} y_\tau^{\gamma}$, for $\tau \in \{A, B\}$, $\tau' \neq \tau$, this implies $Y_\tau^{a,m}\left(y_{\tau'}\right) = \left[p\left(\theta^* - \frac{\eta}{2}\right) Z_\tau y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$, so $y_\tau^{a,m} = \left[p\left(\theta^* - \frac{\eta}{2}\right) Z_{\tau'}^{\frac{1}{2\gamma+2}} Z_\tau^{\frac{2\gamma+1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$. Thus, $V_A^{a,m} = V_B^{a,m} = \left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{1+\gamma}{\gamma}}\left[Z_{\tau'} Z_\tau\right]^{\frac{1}{2\gamma}}$. ■

Proof of Theorem 9. If entrepreneur τ does not expect entrepreneur τ' to innovate, she innovates iff $k_\tau \leq \underline{k}_\tau^{a,m} \equiv q_\tau \mathcal{U}_\tau^{a,SF}$, the same cutoff as without the possibility of a merger. However, if entrepreneur τ expects entrepreneur τ' to innovate, she innovates iff $k_\tau \leq \bar{k}_\tau^{a,m} \equiv q_\tau q_{\tau'} \Upsilon_\tau^{a,m} + q_\tau\left(1 - q_{\tau'}\right)\mathcal{U}_\tau^{a,SF}$. Because $\Upsilon_\tau^{a,a} = \mathcal{U}_\tau^{a,SS} + \frac{1}{2}\left(\mathcal{U}^{a,m} - \mathcal{U}_A^{a,SS} - \mathcal{U}_B^{a,SS}\right)$, if $\Upsilon_\tau^{a,m} > \mathcal{U}_\tau^{a,SS}$, $\bar{k}_\tau^{a,m} > \bar{k}_\tau^{a}$, so the cutoff will be larger when mergers are possible, resulting in more innovation. Thus, it is sufficient to show that $\mathcal{U}^{a,m} > \mathcal{U}_A^{a,SS} + \mathcal{U}_B^{a,SS}$.

Because $V^{a,m} = V_A^{a,m} + V_B^{a,m}$, the merged firm earns $\mathcal{U}^{a,m} = 2\frac{\gamma}{1+\gamma}\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{1+\gamma}{\gamma}}\left[Z_A Z_B\right]^{\frac{1}{2\gamma}}$. Each entrepreneur could earn $\mathcal{U}_\tau^{a,SS} = \frac{1}{2^{\frac{1}{\gamma}}}\left[p\left(\theta^* - \frac{\eta}{2}\right)\right]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{2\gamma}} Z_{\tau'}^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}$ if they did not merge, so $\mathcal{U}_A^{a,SS} + \mathcal{U}_B^{a,SS} = \mathcal{U}^{a,m}\frac{1}{2^{\frac{1}{\gamma}}}\frac{2\gamma+1}{2\gamma}$. Because[43] $\frac{1}{2^{\frac{1}{\gamma}}}\frac{2\gamma+1}{2\gamma} \in (0,1)$ for $\gamma \in (0,\infty)$, merger adds value: $\mathcal{U}^{a,m} > \mathcal{U}_A^{a,SS} + \mathcal{U}_B^{a,SS}$, so $\Upsilon_\tau > \mathcal{U}_\tau^{a,SS}$. ■

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[42] If $\frac{Z_\tau}{Z_{\tau'}} \in \left(\psi, e^{\eta(\gamma+1)}\right)$, there is a mixed strategy equilibrium: one firm selects $y_{\tau'} = \underline{y}_{\tau'}$ and the other randomizes between $y_\tau^{a,SF}$ and $Y_\tau^{a,SS}\left(y_{\tau'}\right)$.

[43] Define $x = \frac{1}{\gamma}$, and $f(x) = 2^{-x-1}(2+x)$: $f'(x) = 2^{-x-1}\left[1 - (2+x)\ln 2\right]$, which is strictly negative because $2\ln 2 > 1$, $\lim_{x\to 0^+} f(x) = 1$, and $\lim_{x\to+\infty} f(x) = 0$. Therefore, $\frac{1}{2^{\frac{1}{\gamma}}}\frac{2\gamma+1}{2\gamma} \in (0,1)$ for all $\gamma \in (0,\infty)$.

B Appendix: Demand Uncertainty

A key ingredient of our paper is that program (1) is a strictly convex programming problem which generates "interior beliefs" for well-diversified portfolios. In the main body of the paper, the possibility of such interior beliefs is a consequence of (strict) convexity of the relative entropy function $R(\cdot)$, which produces a strictly convex core beliefs set $\mathcal{M}$ (see Figure 1). Thus, no specific parametric restriction on the joint probability p is needed to generate our results. In this appendix, we present an alternative "micro-foundation" where interior beliefs are the outcome of uncertainty about consumer demand. All results in our paper remain qualitatively the same in this specification.

Consider a simple extension of our three-dates model. There are three types of goods: type τ goods, $\tau \in \{A, B\}$, and the numeraire. There are two firms, each specializing in the production of goods of type τ. At $t = 1$, entrepreneurs decide whether to pay the discovery cost to innovate. If successful, at $t = 2$, each entrepreneur will select the optimal investment into the project, y_τ, financed by issuing equity to uncertainty-averse investors. The investment decision is made under demand uncertainty for each product (as described below). At $t = 3$, consumer demand is revealed and production decisions of firms are made. If successful, entrepreneurs will be monopolists in their innovative good market. For tractability, we assume that entrepreneur τ has production costs $c_\tau(Q_\tau) = K_\tau Q_\tau$, and that the intermediate investment y_τ lowers, at a cost $\xi(y_\tau) = \frac{\kappa}{2} y_\tau^2$, the per-unit production cost: $K_\tau = K_0 - K_1 y_\tau$.

There are two types of consumers, type A and type B, with a total mass of 1. Consumers value both goods, as well as the numeraire, but each consumer values one good more that the other, which determines their type. The price of the numeraire is fixed to 1, while the price of type τ good, P_τ, is determined in equilibrium. For simplicity, we assume quadratic utility for each type of consumer. Thus

$$U^\tau\left(q_\tau^\tau, q_{\tau'}^\tau\right) = (D + \Delta)\, q_\tau^\tau - \frac{\beta}{2}\left(q_\tau^\tau\right)^2 + D q_{\tau'}^\tau - \frac{\beta}{2}\left(q_{\tau'}^\tau\right)^2 + w - P_\tau q_\tau^\tau - P_{\tau'} q_{\tau'}^\tau,$$

where D, Δ, and β are strictly positive parameters. For simplicity, we assume that w and D large enough so that consumers (in equilibrium) always consume a positive amount of all goods available in the market. It is easy to verify that the consumer τ's demand function for good τ is $q_\tau^\tau = \frac{1}{\beta}(D + \Delta - P_\tau)$, and for good τ' is $q_{\tau'}^\tau = \frac{1}{\beta}(D - P_{\tau'})$. Let $m_\tau \in [m_L, m_H]$ be the proportion of consumers of type τ, with $m_A + m_B = 1$. Market clearing condition for good τ requires that $m_\tau q_\tau^\tau + m_{\tau'} q_\tau^{\tau'} = Q_\tau$, where Q_τ is the output of a firm type τ. Thus, market clearing requires that

$$P_\tau(Q_\tau) = D + m_\tau \Delta - \beta Q_\tau,$$

and the price of type-τ goods is increasing in m_τ. Because producers know m_τ when making their production decisions Q_τ, they maximize

$$\pi_\tau(Q_\tau) = P_\tau(Q_\tau)\, Q_\tau - K_\tau Q_\tau,$$

which gives

$$Q_\tau = \frac{D + m_\tau \Delta - K_\tau}{2\beta}.$$

Letting $\Pi_\tau = \max_{Q_\tau} \pi(Q_\tau)$, we have that entrepreneur τ profits are

$$\Pi_\tau = \frac{\left[D + m_\tau \Delta - K_\tau\right]^2}{4\beta}.$$

This implies that, when both entrepreneurs are successful, investors beliefs are determined by solving:

$$\begin{aligned}
\min_{\{m_A, m_B\}} \mathcal{U} \quad &\equiv \quad \omega_A \left[\frac{\left[D + m_A \Delta - K_A(y_A)\right]^2}{4\beta} - V_A\right] + \omega_B \left[\frac{\left[D + m_B \Delta - K_B(y_B)\right]^2}{4\beta} - V_B\right] + \omega_0 \\
s.t. \quad & \quad m_A + m_B = 1,
\end{aligned}$$

which is a (strictly) convex programming problem, with the same qualitative properties as (1).

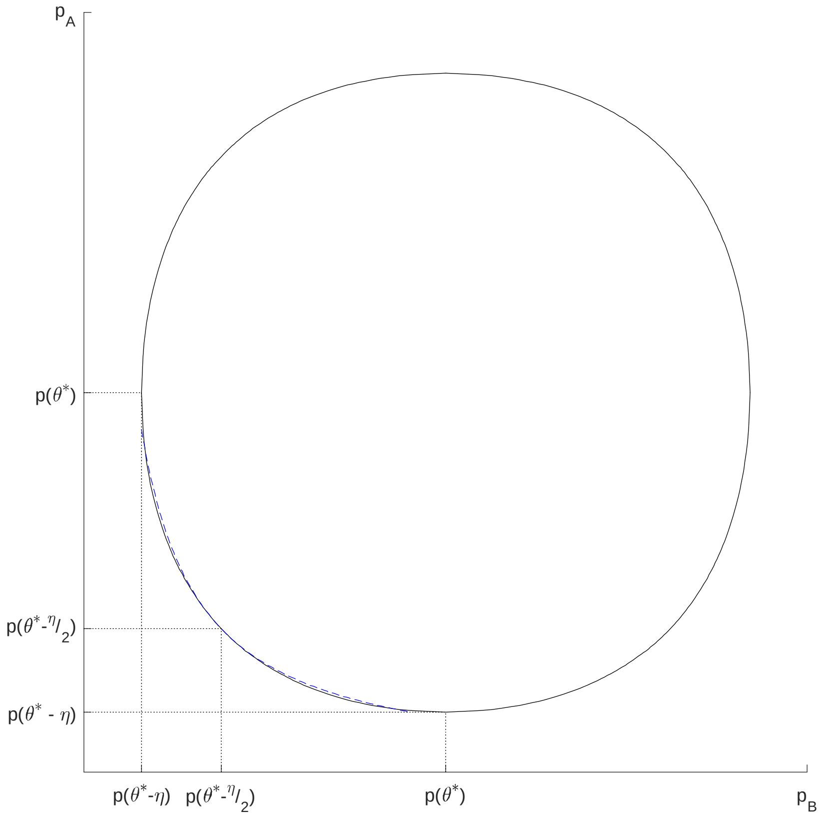

Figure 1: Core Belief Set Under Relative Entropy

This figure displays the core of belief set $\mathcal{M}$ under relative entropy (solid line) given by the set of $p = (p_A, p_B)$ that satisfy $\{p|R(p|\hat{p}) \le \tilde{\eta}\}$ when $\hat{p}_A = \hat{p}_B = \frac{1}{2}$ and $\tilde{\eta} = \frac{3}{5}\ln 2$ (see Table 1). For investors with long positions in both assets, under MEU the relevant portion of the core beliefs set $\mathcal{M}$ is given by the lower-left boundary. The figure also shows the lower-left-hand portion of the specification based on the L^1 norm (in blue and dashed).

Table 1: Numerical examples

1. Base case parameters (Section 1):	$\eta = \frac{3}{5}\ln(2)$	$\tilde{\eta} = 1.7381$	$y_A = y_B = 100$	$c = 3$
	$\hat{p}_A = \hat{p}_B = \frac{1}{2}$	$\theta^* - \theta_M = -0.7954$	$k_\tau = 5$	$q = .75$
2. Uncertainty-neutral case (Section 2.1):	$\bar{k}_\tau = 30.86$		$V_A = V_B = 45.14$	
Uncertainty-averse case (Section 2.2):	$\underline{k}^a_\tau = 3.70$	$\bar{k}^a_\tau = 9.87$		
	$p(\theta^* - \eta) = .0794$	$V^{a,SF}_\tau = 7.94$	$p(\theta^* - \frac{\eta}{2}) = .1893$	$V^{a,SS}_\tau = 18.93$
3. Innovation waves (Section 3):	$\delta = .97$	$\pi = .6$	$c = 3$	$k = 5$
	$\underline{\nu} = 2$	$\nu^e = 7$	$V^a_{nt}(\nu^e) = 34.16$	$V^a_{nt}(1) = 7.70$
	$k_0 = 4.55$	$\underline{k}_d = 0$	$\bar{k}_d = 39.56$	
4. Competition and innovation (Section 4):	$\varepsilon = .01$	$\xi = .1$	$\underline{\xi} = .58$	
	$\underline{\nu}^c = 2$	$\nu^e(\xi, \varepsilon) = 4$	$V^{ac}_{nt}(\nu^e(\xi, \varepsilon)) = 20.67$	$V^{ac}_{nt}(1) = 7.70$
	$\underline{k}^c_d = 0$	$\bar{k}^c_d = 17.14$	$T^e = 6.84$	$R = 0.59$
5. Innovation and acquisitions (Section 5):	$Z_A = Z_B = 1000$	$\gamma = 1$	$y^* = 451.04$	$V^*_\tau = 203.77$
	$y^{a,SF} = 79.38$	$y^{a,SS} = 94.65$	$y^{a,m} = 189.30$	
	$V^{a,SF}_\tau = 6.30$	$V^{a,SS}_\tau = 17.90$	$V^{a,m}_\tau = 35.83$	
	$\bar{k}^a_\tau = 8.15$	$\underline{k}^{a,m}_\tau = \underline{k}^a_\tau = 2.36$	$\bar{k}^{a,m}_\tau = 10.67$	